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The Rational Plane Quartic as Derived from the Norm-Curve in Four Dimensions by Projection and Section.

By J. R. Conner.

§ 1. *Introduction.*

The theory of the rational plane quartic has been a subject of study among mathematicians for years, but although many interesting properties of the curve have been developed, there is still much work to be done before it can be said to be thoroughly known. Veronese[*] has shown that any rational curve of order n may be regarded as a projection of the rational norm-curve of order n in a space of n dimensions; and similarly that any rational curve of class m may be regarded as a section of a developable of the norm-curve of order m in m dimensions. He calls attention to the usefulness of the method of projection and section as an approach to the study of curves in general, and, in particular, to the study of rational curves. The theory of the rational quartic in space has been treated from this point of view by Marletta,[†] who later applied the same method to the study of the rational quintic in space.[‡] Stahl[§] deduced many properties of the rational plane quartic by projection from space, as an introduction to an extended analytic treatment of the curve in the plane. In this paper I shall deduce some of the properties of the rational quartic in the plane from those of the norm-quartic in four dimensions by projection and section. The method of the paper will be in the main synthetic, though analysis will not be entirely dispensed with. It is not to be hoped that, within narrow limits, a thorough treatment of the subject can be given; I shall be satisfied if in a rather sketchy account I succeed in demonstrating the suggestiveness and fruitfulness of the method.

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[*] Veronese, *Math. Ann.*, 19, p. 208.

[†] Marletta, *Annali di Mat.*, Ser. 3, Vol. VIII, p. 97.

[‡] Marletta, *Rend. Palermo*, Vol. XIX, pp. 94–119.

[§] Stahl, *Journal für Math.*, 101, p. 300.

Many of the theorems as stated are known. I have not thought it necessary to give minute references for all of these. A great many references to the literature will be found in the works cited. Much use is made throughout of the properties of the rational quartic in space. An historical account of this curve and references are to be found in a work by Berzolari:* *Sui Combinanti dei sistemi di forme binarie annessi alle curve gobbe razionale del quart' ordine.*

It may be of interest to the casual reader to know just what of novelty may be expected of the paper. The correlation of invariants and a type of covariants of the plane rational quartic in §8 is, in the main, evident from the point of view of the plane, but has not, so far as I know, been pointed out in all its generality. The fact that a number of properties of the rational plane quartic may be made to depend on one fundamental idea, as, for instance, the idea of syzygetic lines and planes (§ 9), seems worthy of note. The three-to-one correspondence of § 11 furnishes an example of a class of geometrical transformations that will, I am sure, be used more and more as their properties become better known.†

I take this opportunity of expressing my indebtedness to Professors Morley and Coble of the Johns Hopkins University, whose suggestions and encouragement have been invaluable in the preparation of this paper. I wish also to express my appreciation of the financial aid of the Carnegie Institution, without which my presence at this University would have been impossible.

§ 2. *Notation.*

In a space or *flat* of four dimensions, F_4, there exist flats of lower dimensions: F_0, F_1, F_2, F_3. The zero-dimensional flat, F_0, is a point; there are ∞^4 of these in F_4. Similarly, there are respectively ∞^6, ∞^6, ∞^4 F_1's, F_2's and F_3's in F_4. We shall frequently use the words point, line, plane and space instead of the symbols F_0, F_1, F_2 and F_3.

There are two general classes of operations with which we shall deal: *projection* and *section.* We may indicate the operation of projection by the Roman letter, F, with a subscript 0, 1 or 2, the subscript indicating the dimensions of the flat from which the projection is made. Similarly we may indicate the operation of section by the Greek letter Φ, the subscript 1, 2 or 3 referring to the dimensions of the flat by which the section is made. We distinguish flats of the same dimension by superscripts as $F_0^{(1)}$, $\Phi_3^{(a)}$. Thus a projection of a curve C from a point 1 on a plane 2 may be indicated by $C\,F_0^{(1)}\,\Phi_2^{(2)}$.

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* Berzolari, *Ann. di Mat.*, Ser. 2, Vol. XX, p. 101.

† Cf. Sturm: *Die Lehre von den geometrischen Verwandtschaften*, Vol. IV, p. 420.

Just as in ordinary space we have systems of ∞^1, ∞^2 and ∞^3 lines, viz., ruled surfaces, congruences, and complexes, so in F_4 we have systems of ∞^1, ∞^2, ∞^3, ∞^4 and ∞^5 lines, and the dual systems of ∞^1, ∞^2, ∞^3, ∞^4 and ∞^5 planes. There are two ways in which we may interpret our operations on these: 1) we may ask for the elements of the system incident with a given flat; or 2) we may ask for the result of the operation with the flat on the elements of the system. The first of these we indicate by a dot (.) preceding the sign of operation. Thus, let K be a system of ∞^5 planes in F_4. $K.\Phi_3^{(\alpha)}$ represents the totality of the planes of the system lying in the space α; these touch a surface in α. $K.F_0^{(a)}\Phi_3^{(\alpha)}$ is a complex of lines in α, the traces on α of the planes of the system K which pass through the point a. $K.F_0^{(a)}\Phi_3^{(\alpha)}.\Phi_2^{(\pi)}$ is the complex curve of this complex on the plane π, if π is in α.

We shall use, in the main, the letters $a, b, c, \ldots, x, y, z, \ldots$, as symbols for points, or, with subscripts, as their coordinates; and the letters $\alpha, \beta, \gamma, \ldots, \xi, \eta, \zeta, \ldots$, in a similar way for spaces.

We shall consider the norm-quartic in F_4 as fixed throughout our discussion, and shall call it R. Its osculating planes are known to lie on a three-way spread of order 6; we shall call this Σ. Dually we shall use the letter S in referring to the two-way locus of tangent lines of R. This is also of order 6.

If no ambiguity can result, the letters F and Φ indicating projection and section will usually be omitted (§ 4).

§ 3. *Certain Spreads Associated with R in F_4.*

It will be well to consider in the first place a few spreads covariantly connected with R. The norm-curve is the ideal medium for the interpretation of the idea of apolarity, and we shall make extensive use of this notion here. Any F_3 in F_4 meets R in a set of four points, thus defining a quartic in the binary domain on R; dually, through any point of F_4 there are four hyperosculating spaces to R, and a point also defines in this way a binary quartic on R. The apolarity-condition of these two quartics is merely the incidence condition of point and space.

We may give R parametrically by

$$x_0 = 1, \quad x_1 = -t, \quad x_2 = t^2, \quad x_3 = -t^3, \quad x_4 = t^4; \tag{1}$$

or, in spaces,

$$\xi_0 = t^4, \quad \xi_1 = 4t^3, \quad \xi_2 = 6t^2, \quad \xi_3 = 4t, \quad \xi_4 = 1. \tag{2}$$

The quartic determined by any point x may be obtained by substituting in the equation of the point, $(\xi x) = 0$, the parametric values of ξ_i as given by (2). This is

$$x_0 t^4 + 4 x_1 t^3 + 6 x_2 t^2 + 4 x_3 t + x_4 \equiv (x t)^{4}{}^{*} = 0, \tag{3}$$

and the quartic determined by any space ξ is

$$\xi_0 - \xi_1 t + \xi_2 t^2 - \xi_3 t^3 + \xi_4 t^4 \equiv (\xi \overleftarrow{t})^{4}{}^{*} = 0. \tag{4}$$

The apolarity-condition of the quartics (3) and (4) is merely the incidence-condition of x and ξ:

$$|\xi x|^4 \equiv \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 = 0. \tag{5}$$

We use the symbol $I_{k,\, n-k}$ for a binary involution of groups of n points in which k given points in general determine uniquely the remaining $n - k$ of a group.

It follows from (5) that the involution, $I_{3,1}$, of binary quartics apolar to the quartic (3) is cut out of R by spaces on x.

An $I_{1,3}$ of quartics

$$(a t)^4 + \lambda (b t)^4 = 0 \tag{6}$$

is determined by the points of the line joining the points a and b; the involution of quartics apolar to all sets of (6) is

$$(\alpha \overleftarrow{t})^4 + \lambda (\beta \overleftarrow{t})^4 + \mu (\gamma \overleftarrow{t})^4 = 0, \tag{7}$$

where α, β, γ are three linearly independent spaces on the line.

If we put any invariant condition on (3), we have a spread in F_4, the locus of points giving quartics on R for which the given invariant vanishes. The quartic has two invariants, g_2 and g_3. We have the corresponding spreads:

$$g_2 \equiv x_0 x_4 - 4 x_1 x_3 + 3 x_2^2 = 0,$$

$$g_3 \equiv \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{vmatrix} = 0.$$

g_2 is a quadric spread in S_4, the locus of points defining self-apolar quartics on R.

If the quartics defined by points x and y are apolar, y must be on the space on the four points defined by x, and conversely. We thus have an involutory

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[*] It is convenient to regard the x's or ξ's in this way as coefficients of binary forms in t, and to use the symbolic notation in the ordinary way in the manipulation of these coefficients. (4) differs from (3) in the way in which the coefficients are associated with the powers of t. The arrowhead is intended to indicate this.

correlation connecting x and y. It is a polarity, *exactly the polarity of the quadric* g_2.* For the apolarity-condition of the quartics $(xt)^4$ and $(yt)^4$ is

$$x_0 y_4 + x_4 y_0 - 4(x_1 y_3 + x_3 y_1) + 6 x_2 y_2 = 0,$$

the polarized form of g_2.

In a polarity in F_4 we have polar point and space, and also polar line and plane, p and π say. π may be regarded as the axis of the pencil of polar spaces of points of p, or as the locus of polar points of spaces on p. Or, polar points and spaces of spaces and points of either p or π are incident with the other. Similarly, polar planes of lines on π contain p, and dually, and polar planes of lines meeting p have a space and a line in common with π (meet π in a line), and dually. Polar lines of planes meeting p meet π, and dually.

Let us next consider g_3: It is the locus of points defining catalectic quartics on R, or the locus of points such that the spaces on them cut out of R *involutions* $I_{3,1}$ *with a neutral pair.* If yz on R is a neutral pair of the involution defined by a point x, then *any* two points on R may be cut out by a space on xyz; there are therefore ∞^2 spaces on xyz, and x, y and z are on a line. x is therefore on a bisecant line of R, and we have:

g_3 *is the locus of lines bisecant to* R.†

Any spread of lines bisecant to a curve in F_4 is only a two-way locus of spaces, the tangent spaces coinciding along a bisecant line. For, let

$$x_i = \phi_i(t) \qquad (i = 0, 1, 2, 3, 4)$$

be a curve. The spread of bisecants is

$$x_i = \phi_i(s) + \lambda \phi_i(t).$$

The tangent F_3 at a point (s, t, λ) is the determinant

$$\left| y_i, \ \frac{d}{ds}\phi_i(s), \ \frac{d}{dt}\phi_i(t), \ \lambda\frac{d}{dt}\phi_i(t), \ \phi_i(s) + \lambda\phi_i(t) \right| = 0,$$

which reduces to

$$\left| y_i, \ \phi_i(s), \ \phi_i(t), \ \frac{d}{ds}\phi_i(s), \ \frac{d}{dt}\phi_i(t) \right| = 0,$$

and as this is independent of λ our statement is proved. In particular, the hyperosculating space of the curve at a given point contains two consecutive pairs of lines of the spread, and hence must touch it along the tangent line to the curve at the point of hyperosculation. g_3 is a two-way of spaces of class 4 as we shall show later.

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* Clifford, *Mathematical Papers*, p. 312.

† Cf. Segre, *Mem. Torino*, Ser. 2, Vol. XXXIX, p. 36.

The spread Σ is the locus of points defining quartics with two equal roots; S is the locus of points defining quartics with three equal roots. Σ is, analytically, the discriminant of $(xt)^4 = 0$; S is the complete intersection of g_2 and g_3. Their orders, 6, are obvious from this.

The quadratic invariant of (4) defines a spread of class 2, γ_2. γ_2 is easily seen to be identical with g_2. The cubic invariant of (4) defines a spread, γ_3, of class 3. It is the dual of g_3, and hence must be a two-way of order 4. We reserve its discussion for a later section.

There are ∞^3 planes trisecant to R, and ∞^2 lines in each; lines which carry trisecant planes to R are therefore in a system of ∞^5 lines, a *hypercomplex*. We shall call this the hypercomplex Q. To determine the order of Q, let us consider the surface $Q . F_0^{(a)} \Phi_3^{(a)}$. This is obviously the locus of lines trisecant to $R F_0^{(a)} \Phi_3^{(a)}$. The latter curve is a rational quartic in α, and its trisecant lines are known to lie on a unique quadric. It follows that the spread of lines of Q on a, $Q . F_0^{(a)}$, is of order 2, and we have

Lines carrying planes trisecant to R are in a quadratic hypercomplex.

The tangent line to R at any point t may be written

$$y_i = x_i + \lambda \frac{dx_i}{dt};$$

or, homogeneously,

$$y_i = \tau_1 \frac{\partial x_i}{\partial t_1} + \tau_2 \frac{\partial x_i}{\partial t_2}, \tag{8}$$

where x_i have the values given in (1). The line (8) meets any space ξ in the point determined by the equation

$$(\xi y) = \tau_1 \frac{\partial (\xi t)^4}{\partial t_1} + \tau_2 \frac{\partial (\xi t)^4}{\partial t_1} = 0,$$

whence we may take

$$\tau_1 = \frac{\partial (\xi t)^4}{\partial t_2}, \quad \tau_2 = - \frac{\partial (\xi t)^4}{\partial t_1},$$

and

$$y_i = \frac{\partial x_i}{\partial t_1} \frac{\partial (\xi t)^4}{\partial t_2} - \frac{\partial x_i}{\partial t_2} \frac{\partial (\xi t)^4}{\partial t_1} = J\,[x_i,\ (\xi t)^4], \tag{9}$$

where we indicate by J the Jacobian of x_i and $(\xi t)^4$.

The equations (9) *are a parametric representation of the rational sextic curve cut out of S by the space ξ; using our operational symbols, it is $S \Phi_3^{(\xi)}$.*

If $(\xi t)^4$ is the point t_1 taken four times, the equations (9) have the common factor $|tt_1|^3$, and (9) thus reduces to a rational cubic. ξ is in this case a hyperosculating space of R, or, as we may say, a space of R. (9) is now only the polarized forms of the values of x as to t_1. We have therefore the theorem:

The first osculants of R are cut out of S by the spaces of R.[*]

The parametric representation of the curve $\Sigma\,\Phi_2^{(\pi)}$ is also easily found. The osculating plane to R at the point t is

$$y_i = \lambda \frac{\partial^2 x_i}{\partial t_1^2} + \mu \frac{\partial^2 x_i}{\partial t_1 \partial t_2} + \nu \frac{\partial^2 x_i}{\partial t_2^2},$$

where λ, μ, ν are homogeneous variable parameters. This meets the plane π which, let us say, is defined by the pencil of spaces

$$(\xi x) + \lambda (\eta x) = 0, \tag{10}$$

in the points given by

$$y_i = \begin{vmatrix} \dfrac{\partial^2 x_i}{\partial t_1^2}, & \dfrac{\partial^2 x_i}{\partial t_1 \partial t_2}, & \dfrac{\partial^2 x_i}{\partial t_2^2} \\ \dfrac{\partial^2 \xi}{\partial t_1^2}, & \dfrac{\partial^2 \xi}{\partial t_1 \partial t_2}, & \dfrac{\partial^2 \xi}{\partial t_2^2} \\ \dfrac{\partial^2 \eta}{\partial t_1^2}, & \dfrac{\partial^2 \eta}{\partial t_1 \partial t_2}, & \dfrac{\partial^2 \eta}{\partial t_2^2} \end{vmatrix}, \tag{11}$$

where we have written ξ for $(\xi t)^4$ and η for $(\eta t)^4$. A curve (11) is evidently completely defined as soon as we give the pencil of spaces (10) or the pencil of binary quartics on R which it defines.

§ 4. *Section by a Space.*

In the operations of projection with which we shall deal only points and lines are involved as the elements from which we project; similarly in the operations of section we are concerned with planes and spaces as the elements by which the operation is determined. No ambiguity can result if we omit the symbols of projection and section, F and Φ, and write merely the symbol of the element involved in the operation; thus $Rp\pi$ is the result of projecting R from a line p on a plane π. The dot (.) will be used in the sense previously explained to indicate that we are asking for the elements of a given system of planes or lines incident with the element by which the operation is determined.

The curve $S\alpha$ is the dual of $Ra\alpha$. $S\alpha$ is a curve in the space α of order 6 and class 4; it is a rational quartic of planes. Its four cusps are obviously the four points $R\alpha$. It is the complete intersection of the cubic surface $g_3\alpha$ and the quadric surface $g_2\alpha$. Let b_t, p_t, π_t, β_t be the point, plane, line, and space, respectively, of R at the point t. Then $p_t\alpha$ is a point of $S\alpha$; we may name this also with the parameter t. $\pi_t\alpha$ and $\beta_t\alpha$ are the line and plane of $S\alpha$ at the point t. $g_2\alpha$ and $g_3\alpha$ are the loci of points whose four planes of $S\alpha$ give quartics respectively self-apolar and catalectic. Since β_t touches g_3 along the line p_t, it follows that $\beta_t\alpha$, the plane of $S\alpha$ at the point t, touches $g_3\alpha$ at this point; *i. e.*, $S\alpha$ is an asymptotic curve on $g_3\alpha$.*

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[*] Cf. Berzolari, *Annali di Mat.*, Ser. 2, Vol. XXI, p. 7.

We have seen that the cubic osculants of R are the curves $S\beta_t$. If a rational curve is projected into a lower space, its osculants are projected into the osculants of the new curve. It follows that the osculants of $Ra\alpha$ are the curves $S\beta_t a\alpha$.

Dually, the cubic osculant t of $S\alpha$ is the envelope of the system of planes $\pi_{t'} b_t\alpha$; in other words, the envelope of the system of planes into which the planes $\pi_{t'}$ of R are projected from a point t of R. It follows:

The osculants of $S\alpha$ *considered as point-loci are the curves* $Rb_t\alpha$.

Rb_t is a cubic cone projecting R from the point b_t of it. It follows that Rb_t is on g_3. Or:

$g_3\alpha$ *is the locus of the curves* $Rb_t\alpha$, *the cubic osculant of* $S\alpha$ *considered as point-loci.*†

It is easily seen that $Rb_t\alpha$ is on the cusps of $S\alpha$, touches $S\alpha$ at the point t and has the same osculating plane with $S\alpha$ at t. Two osculants $Rb_t\alpha$ and $Rb_{t'}\alpha$ meet in one and only one point of α apart from the four points $R\alpha$, namely, the point $b_t b_{t'}\alpha$. Three osculants of $S\alpha$, $Rb_t\alpha$, $Rb_{t'}\alpha$, $Rb_{t''}\alpha$, meet, two by two, in three points of a line, the line $b_t b_{t'} b_{t''}\alpha$. Now $b_t b_{t'} b_{t''}$ is a plane trisecant to R, and $b_t b_{t'} b_{t''}\alpha$ is a line on this plane. This gives us an interpretation of the complex $Q.\alpha$ in α:

The three points of intersection, two by two, of three osculants of $S\alpha$ *are on a line. Such lines are in the quadratic complex* $Q.\alpha$.

The planes of the tetrahedron $R\alpha$ are singular planes of $Q.\alpha$, since obviously every line on one of these planes carries a trisecant plane to R. Hence:

$Q.\alpha$ *is a tetrahedral complex; the points* $R\alpha$ *are the vertices of the fundamental tetrahedron.*

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* Segre, *loc. cit.*

† Study, *Leipz. Berichte*, 1886, p. 9.

If π is a plane in α the conic $Q.\pi$ touches the lines cut out of the faces of this tetrahedron by π.

$g_3\alpha$ contains the six lines joining the four vertices of this tetrahedron, and is therefore a four-nodal cubic surface, with the points $R\alpha$ for nodes.* The conic $Q.\pi$ therefore touches four lines whose vertices are on the cubic curve $g_3\pi$. There must be an infinity of such four-lines, one defined by every space α on π.

The four tangent planes from a line of α to $g_3\alpha$ are on the tangent spaces from this line to g_3. It follows:

g_3, *in spaces, is a two-way of class* 4.

Tangent spaces to g_3 may be regarded as spaces bitangent to R. Dually we have:

The double spread of Σ *is a two-way,* γ_3, *of order* 4, *and a three-way of class* 3.

γ_3 may be regarded as the locus of points of intersection of pairs of planes of R.

§ 5. *Mapping of $g_3\alpha$ on a Plane.*

Some light is shed on the figures with which we are dealing by the mapping of $g_3\alpha$ on a plane, π. We can represent plane sections of $g_3\alpha$ by cubics of the threefold linear system Ψ on the six points of intersection of four lines 1, 2, 3, 4 in π. The six points may be named 12, 13, 14, 23, 24, 34. Now any cubic of Ψ meets any of the four lines, 1 say, in three points 12, 13, 14, and hence, if a cubic Ψ is forced on another point of this line, it must degenerate into 1 and a conic on 23, 34, 42. In other words, a plane section of $g_3\alpha$ on the map of *any* point of 1 is on the map of *every* point of 1, and it follows that 1 is represented by a single point of $g_3\alpha$. That this point is a node of $g_3\alpha$ follows from the fact that conics on 34, 42, 23 meet 1 twice, and hence every plane section on the point 1 of $g_3\alpha$ passes twice through that point. 1, 2, 3, 4, therefore, map into nodes of $g_3\alpha$. The points 12, etc., give the six lines joining the nodes. The lines of the diagonal triangle of 1, 2, 3, 4 map into three lines on a plane, the three further lines of $g_3\alpha$. These lines are important and use will be made of them later.

Consider three points x, y, z of π which, with the six points 12, etc., are base-points of a pencil of cubics. Such a set of three points maps into the three points of intersection of $g_3\alpha$ with a line p in α. Now x, y, z must be on a conic with 23, 34, 42, the map of this conic on $g_3\alpha$ being the plane section joining the node 1 to the line p. It follows that 2, 3, 4, xy, yz, zx are six lines of a conic, C. By a similar argument 1, 2, 3, xy, yz, zx are six lines of a conic, necessarily the same conic, C. xy, yz, zx therefore touch a conic of the range on 1, 2, 3, 4. Therefore we have

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* Segre, *loc. cit.*

The three meets of $g_3\alpha$ with a line are maps of the vertices of a triangle circumscribed to a conic of the range on 1, 2, 3, 4.

It follows easily from this that the maps of the conics of this range are the asymptotic curves of $g_3\alpha$. $S\alpha$ is therefore the map of a conic, C, of the range on 1, 2, 3, 4. The points of contact of C with 1, 2, 3, 4 give the cusps of $S\alpha$.

A first osculant of $S\alpha$ is a space cubic on the cusps $R\alpha$ and touching $S\alpha$ at a definite point, being thereby uniquely determined. Hence:

First osculants of $S\alpha$ are the maps of tangents to C. The tetrahedral complex $Q\,.\,\alpha$ is represented on π by triangles circumscribed to C, the vertices of such a triangle mapping into the three points in which a line of $Q\,.\,\alpha$ meets $g_3\alpha$.

This last theorem is the object of this section; the representation of the four-nodal cubic surface on a plane is well-known.

§ 6. *The Curve $\Sigma\pi$.*

$\Sigma\pi$ is the dual of $Rp\pi$, and is a rational curve of order 6 and class 4. If β_t is the space t of R, we call the line $\beta_t\pi$ the line t of $\Sigma\pi$. The point t of $\Sigma\pi$ is $\pi_t\pi$, π_t being the plane of R at the point t. $S\pi$ is a group of six points, the intersections of $g_2\pi$ and $g_3\pi$. From a point of the group $S\pi$ three spaces to R and hence three tangents to $\Sigma\pi$ coincide. The points $S\pi$ are therefore cusps of $\Sigma\pi$.

Spaces on π meet R in sets of a pencil $(I_{1,3})$ of quartics

$$(\alpha t)^4 + \lambda(\beta t)^4 = 0, \tag{1}$$

where α and β are two independent spaces on π. The space $\alpha + \lambda\beta = 0$ is incident with every point of π. It follows that all quartics of the pencil (1) are apolar to all quartics determined on R, or $\Sigma\pi$, by points of π, and therefore (1) is the fundamental involution of $\Sigma\pi$. Hence:

Quartics of the fundamental involution of $\Sigma\pi$ are cut out of R by spaces on π.

It will be of advantage in the study of the curve $\Sigma\pi$ to consider the spreads determined by R on a space α on π. It should be remembered, however, that in this way we isolate a set of the fundamental involution on $\Sigma\pi$, and hence all curves thus obtainable on π are not intrinsic to $\Sigma\pi$ alone. We have considered in a previous section some of the geometrical forms determined by R on a space α.

$\Sigma\alpha$, the developable of $S\alpha$, meets π in $\Sigma\pi$. The cusps of $\Sigma\pi$ are the six points where $S\alpha$ meets π. The osculants of $\Sigma\pi$ are the intersections with π of the developables of the osculants of $S\alpha$. We see at once that

The curve $g_3\pi$ *may be regarded as the locus of cusps of cubic osculants of* $\Sigma\pi$.

There are a pair of lines of $Q.\alpha$ through any point, x, of $g_3\pi$ and on π; they touch the conic $Q.\pi$. Now we have shown that the lines $Q.\alpha$ are lines bisecant to cubic osculants of $S\alpha$; these two lines therefore pass through the two further cusps of the osculant to $\Sigma\pi$ which has a cusp at x. It follows that

The conic $Q.\pi$ *is the locus of cusp three-lines of cubic osculants of* $\Sigma\pi$, *or* $Q.\pi$ *is the Stahl conic* K *of* $\Sigma\pi$.*

The determination given above for cubic osculants of $\Sigma\pi$ from the point-forms of the osculants of $S\alpha$ does not apply if a_t is a point of the set $R\alpha$. But, regarding the cubic osculants of $S\alpha$ as the systems of planes $\pi_{t'}a_t\alpha$ for varying t', $\pi_{t'}$ and a_t being plane and point of R at t' and t respectively, it is obvious that the osculant at a point t of the set $R\alpha$ is the quartic cone with three cuspidal generators which projects $S\alpha$ from the point t. We thus obtain proper osculants of $\Sigma\pi$ for the points corresponding to $R\alpha$, as we should. The grouping of the cusps of the four osculants to $\Sigma\pi$ at four points of a set of the fundamental involution and the one-to-one relation of the conic K to $\Sigma\pi$, are geometrically obvious here.

The hypercomplex dual to Q is a quadratic system of ∞^5 planes, Q' say. There is a double infinity of planes of Q' on any space α. They touch a quadric which we may call $Q'.\alpha$. This quadric is the locus of lines carrying three spaces of R or three planes of $S\alpha$. $Q'.\alpha$ is therefore the unique quadric touched by planes of $S\alpha$. If

$$(\alpha t)^4 = 0$$

is a quartic giving the four points $R\alpha$, then $(\alpha t)^4$ is a set of the fundamental involution of $\Sigma\pi$; the sets of three planes of $S\alpha$ on lines of $Q'.\alpha$ define, as is known, on $S\alpha$ the apolar cubics of $(\alpha t)^4 = 0$. On any point of the conic $Q.\alpha\pi$ there are four planes of $S\alpha$ and their traces on π give the four lines of $\Sigma\pi$ on this point. Three of these planes meet in a line of $Q'.\alpha$. By means of the fourth plane $Q' \,.\, \alpha\pi$ is in one-one correspondence with $\Sigma\pi$ and the point of $Q' \,.\, \alpha\pi$ is incident with its corresponding line of $\Sigma\pi$. $Q' \,.\, \alpha\pi$ is therefore a *perspective conic* of $\Sigma\pi$, that particular perspective conic associated with the definite set $(\alpha t)^4 = 0$ of the fundamental involution of $\Sigma\pi$.* It may be defined as the locus of points in π three of whose tangents to $\Sigma\pi$ define an apolar cubic of $(\alpha t)^4$.

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* Stahl, *Journal für Math.*, 101, p. 304.

We have said that the two-way γ_3, the double spread of Σ, is of order 4 and class 3. It follows that the surface $\gamma_3 a\alpha$ is a surface of order 4 and class 3, a Steiner quartic surface. Every space section of γ_3 must be a rational quartic since every plane section of a Steiner surface is a rational quartic. The curve $\gamma_3\alpha$ is the double curve of $\Sigma\alpha$, the developable of $S\alpha$. $\gamma_3\alpha\pi$, or merely $\gamma_3\pi$, are the four double points of $\Sigma\pi$.

The quartic curve $\gamma_3\alpha$ lies on a unique quadric surface in α. This quadric surface meets π in a conic on the four nodes of $\Sigma\pi$, and we shall see later that it cuts out of $\Sigma\pi$ in addition the set of the fundamental involution associated with the space α on π. It may be regarded as the locus of lines in α trisecant to γ_3. We see thus that planes on lines trisecant to γ_3 are in a quadratic system. This system and its dual hypercomplex of lines are important and we shall denote a later section to their consideration.

§ 7. *Projection of R from a Line.*

The projection of R from a line p on a plane π, in our notation the curve $Rp\pi$, is a projectively general rational quartic on the plane π. Using a former notation for points and lines of R, the point $a_t p\pi$ is the point t of $Rp\pi$ and the line $p_t p\pi$ is the tangent to $Rp\pi$ at this point.

Spaces on p meet π in lines. They cut out of R the sets of parameters of intersections of lines with $Rp\pi$. The fundamental involution of $Rp\pi$ is defined by the scheme of points common to all of these spaces; that is, by the points of p. For convenience we shall consider π as the polar plane of p as to g_2 or as to R. We have then that the fundamental involution of $Rp\pi$ is defined either by points of p or by spaces on π. Dually the fundamental involution of $\Sigma\pi$ is defined by the same scheme of points or spaces. The

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*Two plane curves, one given in lines and the other in points, are said to be *perspective* if they are in one-one correspondence and corresponding point and line are incident. See Stahl, *Math. Ann.*, 38, pp. 561 and 575. A rational plane quartic has ∞^3 perspective cubics, one being determined uniquely by an arbitrary binary cubic; it has ∞^1 perspective conics, one associated with each set of the fundamental involution. Compare Coble, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, p. 352.

involution defined on $\Sigma\pi$ by tangents through a point of π is determined on R either by points of π or by spaces on p.

The polar point of any space on p as to g_2 is on π. The polar point of the space joining p to any line p' of π is the polar point of p' as to $g_2\pi$. If p' touches $Rp\pi$, two roots of the quartic cut out of R by the space pp' coincide. The polar point of the space pp' must therefore be on Σ, since Σ is the locus of points of F_4 defining quartics on R with a double root. It follows:

The curves $Rp\pi$ and $\Sigma\pi$ are polar reciprocals as to the conic $g_2\pi$.

$g_2\pi$ is the locus of points defining self-apolar quartics on $\Sigma\pi$ and the locus of lines cutting out self-apolar quartics from $Rp\pi$. It touches the stationary lines of $Rp\pi$ and is on the stationary points of $\Sigma\pi$.

An interesting system of configurations is determined on π by projecting from p the six lines joining the four points $R\alpha$ where α is a space on π. The four planes on the points $R\alpha$ meet π in lines of the Stahl conic K of $\Sigma\pi$. The six lines on $R\alpha$ meet π in the intersections of these four tangents to this conic; these are points of $g_3\pi$. Projecting from p, the points $R\alpha p\pi$ are a set of the fundamental involution on $Rp\pi$. The projection gives us the following theorem:

The six lines joining the four points of a set of the fundamental involution on $Rp\pi$ meet $g_3\pi$ in six points of a four-line. These four lines touch the Stahl conic K of $\Sigma\pi$, determining on the latter conic, which is in natural one-one correspondence with $Rp\pi$, the set of the fundamental involution.

We may point out that the curves K and $g_3\pi$ are the polar reciprocals of the dual curves as determined by $Rp\pi$ as to $g_2\pi$.

The locus of lines in F_4 joining points in which spaces on a given plane π meet R is a ruled two-way of order 9. For any one of the varying spaces meets this two-way in the curve $g_3\pi$ and the six lines. This nonic two-way is elliptic, since its lines are in one-one correspondence with $g_3\pi$. Its lines project from p on π into the lines of a curve of class 9. Hence:

The lines of complete four-points of sets of the fundamental involution on $Rp\pi$ touch a curve of class 9 and genus 1.

We may also state the more general theorem:

Any involution

$$(\gamma t)^4 + \lambda(\delta t)^4 = 0$$

on $Rp\pi'$ determines in the same way a cubic and a conic bearing a relation similar *to the above to the four-points of the involution. Lines of complete four-points of the involution on $Rp\pi$ touch a curve of class* 9 *and genus* 1.

For this involution determines a definite plane π'. Using this as the plane in $Rp\pi'$ we have the theorem as we have stated it.

§ 8. *Connection of Invariants of $Rp\pi$ and a Certain Type of Covariants of $\Sigma\pi$.*

Let α, β, γ be three independent spaces on the line p from which we are projecting. We may at once take as coordinates in the plane π

$$\left.\begin{aligned} X_0 &= (\alpha x), \\ X_1 &= (\beta x), \\ X_2 &= (\gamma x). \end{aligned}\right\} \tag{1}$$

The quartic $Rp\pi$ is then

$$\left.\begin{aligned} X_0 &= (\alpha t)^4, \\ X_1 &= (\beta t)^4, \\ X_2 &= (\gamma t)^4.{}^{*} \end{aligned}\right\} \tag{2}$$

The projection of any point of F_4 from p on π is given at once by substitution of its coordinates in (1). We define the coordinates of the line in π joining any two points X, Y as the determinants of the matrix

$$\left\| \begin{matrix} X_0, & X_1, & X_2 \\ Y_0, & Y_1, & Y_2 \end{matrix} \right\|, \tag{3}$$

whence the projection on π of the line

$$p_{ij} \equiv x_i y_j - x_j y_i$$

is

$$\left.\begin{aligned} \Xi_0 &= \Sigma \mid \beta\gamma \mid_{ij} p_{ij}, \\ \Xi_1 &= \Sigma \mid \gamma\alpha \mid_{ij} p_{ij}, \\ \Xi_2 &= \Sigma \mid \alpha\beta \mid_{ij} p_{ij} \end{aligned}\right\} \tag{4}$$

from (1) and (3), where we have written

$$\mid \beta\gamma \mid_{ij} = \beta_i \gamma_j - \beta_j \gamma_i,$$

and Σ is used as a summation-sign. If the line in F_4 is given by three spaces ξ, η, ζ on it, we have

$$\rho p_{ij} = \pi_{klm} = \begin{vmatrix} \xi_k & \xi_l & \xi_m \\ \eta_k & \eta_l & \eta_m \\ \zeta_k & \zeta_l & \zeta_m \end{vmatrix},$$

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* Cf. Veronese, *Math. Ann.*, 19, *loc. cit.*

and (4) becomes

$$\Xi_0 = \begin{vmatrix} \beta_0, & \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ \gamma_0, & \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \\ \xi_0, & \xi_1, & \xi_2, & \xi_3, & \xi_4 \\ \eta_0, & \eta_1, & \eta_2, & \eta_3, & \eta_4 \\ \zeta_0, & \zeta_1, & \zeta_2, & \zeta_3, & \zeta_4 \end{vmatrix}, \tag{5}$$

with similar expressions for Ξ_1 and Ξ_2.

If we require an invariant of $Rp\pi$ to vanish, we are obviously imposing a single condition on the line p from which we are projecting R. Now, if a single condition is imposed on a line in F_4, the line is in a hypercomplex. We have then,

Invariants of $Rp\pi$ are represented in F_4 by hypercomplexes of lines associated with R.

Any hypercomplex in F_4 is a function of determinants of the type

$$| \xi_k, \eta_l, \zeta_m |,$$

where ξ, η, ζ are spaces on a line. In the plane π this of course is equivalent to the theorem that any invariant of a rational curve is a function of determinants of the matrix of coefficients of the binary forms giving the parametric representation of the curve. Any invariant of $Rp\pi$, in our notation, is a function of determinants of the type

$$| \alpha_j, \beta_k, \gamma_l |,$$

such a function defining a hypercomplex in F_4. Now by means of the polarity associated with R, or, which is the same thing, by means of the polarity of the quadric g_2, we have associated with every hypercomplex of lines in F_4 a hypercomplex of planes; that is, a system of planes satisfying one condition. Now every line in F_4 carries ∞^2 planes; if we ask for the planes on a line that lie in a hypercomplex of planes, we have only ∞^1 planes; and the locus of these planes is a cone-spread of the same order as the hypercomplex, and having the line as vertex. Every hypercomplex of planes in F_4 covariantly associated with R obviously determines a covariant curve of $Rp\pi$ when we project from p, the planes of the hypercomplex on p tracing out on π the covariant curve of $Rp\pi$ in question.

We now show how, given a hypercomplex of planes in F_4, we may obtain the equation of the corresponding covariant of $Rp\pi$. Let

$$F(| z_k, y_l, x_m |) = 0$$

be the equation of any covariant hypercomplex in F_4. If a and b are any two points on the line p, the equation of the hypercomplex cone determined by p is

$$F(|\,a_k,\ b_l,\ x_m\,|) = 0.$$

Let α, β, γ be three independent spaces on p. Then

$$\begin{aligned} (\alpha a) &= 0, & (\alpha b) &= 0, & \rho X_0 &= (\alpha x), \\ (\beta a) &= 0, & (\beta b) &= 0, & \rho X_1 &= (\beta x), \\ (\gamma a) &= 0, & (\gamma b) &= 0, & \rho X_2 &= (\gamma x). \end{aligned}$$

These nine equations are all possible bilinear relations among the rows of the two matrices

$$\left\| \begin{matrix} \alpha_0, & \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & X_0 \\ \beta_0, & \beta_1, & \beta_2, & \beta_3, & \beta_4, & X_1 \\ \gamma_0, & \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4, & X_2 \end{matrix} \right\|$$

and

$$\left\| \begin{matrix} a_0, & a_1, & a_2, & a_3, & a_4, & 0 \\ b_0, & b_1, & b_2, & b_3, & b_4, & 0 \\ x_0, & x_1 & x_2, & x_3, & x_4, & -\rho \end{matrix} \right\|,$$

whence follows the proportionality of the corresponding determinants

$$\sigma\,|\,a_k,\ b_l,\ x_m\,| = |\,\alpha_i,\ \alpha_j,\ X\,|, \tag{6}$$

where σ is a proportionality factor, and we have

$$F(|\,a_k,\ b_l,\ x_m\,|) = MF(|\,\alpha_i,\ \alpha_j,\ X\,|), \tag{7}$$

where M is merely a power of the σ in (6) and does not vanish.

The left-hand member of (7) is an invariant of the section of Σ by the plane abx; *i. e.*, it is an invariant of the quartic

$$\begin{aligned} \Xi_0 &= (a t)^4, \\ \Xi_1 &= (b t)^4, \\ \Xi_2 &= (x t)^4, \end{aligned}$$

a quartic given with binomial coefficients. Hence, we have the following:

If, for a rational quartic given without binomial coefficients, we have a covariant curve whose equation is expressible in terms of determinants of the type

$$|\,\alpha_i\ \alpha_j\ X\,|,$$

and we replace each such determinant by the complementary determinant

$$|\,a_k\ b_l\ c_m\,|,$$

where the a, b, c are coefficients of binary quartics written with binomial coefficients *which give the parametric representation of a second rational quartic, we obtain an invariant of the latter quartic.*

We have thus a one-one correspondence between covariants of this special type and invariants of $Rp\pi$. It is easy to discover, geometrically, the corresponding covariant of some of the simpler invariants.

We give in the following table some invariants and corresponding covariants, where both quartics are understood as quartic point-loci. Some of the covariants are more easily defined as line-loci, but they are of course obtained as point-loci from invariants by the above translation principle.

Invariants.	*Covariants.*
I. Triple point condition.	Stahl conic K.
II. Undulation condition.	Quartic (in points).
III. Cusp condition.	Product of stationary lines.
IV. Tac-node condition.	Locus of lines cutting out harmonic pairs from the quartic.
V. Condition that two self-apolar sets of fundamental involution coincide.	Locus of lines cutting out self-apolar quartics from the quartic curve.
VI. Condition of self-apolarity of sextic giving the points of inflection.	Locus of points from which six tangents define a self-apolar sextic.
VII. Condition that two points of inflection belong to a set of the fundamental involution (discriminant of the conic on the points of inflection).	Product of the four double-tangents.
VIII. Flecnode condition.	Locus of points of inflection of cubic osculants.
IX. Condition that three double tangents be on a point.	Conic on the six points of inflection.

This list might be extended indefinitely. It will be obvious from the proofs of these facts that a geometrical characterization of an invariant is, in a sense. a definition of the corresponding covariant.

For convenience the proofs are given for the dual quartics $Rp\pi$ and $\Sigma\pi$. An invariant of $Rp\pi$ defines a hypercomplex C. The curve $C.\pi$ is the corresponding covariant curve of $\Sigma\pi$.

I. The three nodes of $Rp\pi$ are determined by the three lines of g_3 which meet p. If $Rp\pi$ have a triple point, p must carry a plane trisecant to R. The hypercomplex of lines thus defined gives on π the Stahl conic K of $\Sigma\pi$.

II. If $Rp\pi$ have an undulation, p is on a space of R. These spaces, as we have seen, mark out on π the lines of $\Sigma\pi$.

III. If $Rp\pi$ have a cusp, p meets S. Lines in π meeting S are lines through one of the six cusps of $\Sigma\pi$.

IV. If $Rp\pi$ have a tac-node, two nodes have come together and p touches g_3. The locus of lines in π which touch g_3 is the curve $g_3\pi$ taken in lines.

V. If the line p touches g_2, the two self-apolar sets of the fundamental involution of $Rp\pi$ coincide, these being defined by the two points in which p meets g_2. The corresponding curve in π is $g_2\pi$ taken in lines.

VI. The points of inflection of $Rp\pi$ are defined on R by the six points in which p meets Σ. If the six points thus determined on R by p are a self-apolar sextic, the sextic giving the points of inflection of $Rp\pi$ must be self-apolar. The corresponding curve in π is the locus of lines meeting $\Sigma\pi$ in six points whose parameters are a self-apolar sextic.

VII. If two points of inflection belong to one set of the fundamental involution, p meets γ_3. We shall see in § 9 that the conic on the points of inflection is then two lines. The corresponding curve in π is the product of the four points in which γ_3 meets π, the four nodes of $\Sigma\pi$.

VIII and IX will be proved in §§ 10 and 9 respectively.

§ 9. *Syzygetic Lines and Planes and the Associated Hypercomplexes.*

Any point of F_4 determines a quartic on R. The Hessian of this quartic determines another point of F_4. If these two quartics are f and h, its Hessian, any point of the line joining them defines a quartic on R of the form

$$f + \lambda h = 0.$$

Since the Hessian of $f + \lambda h$ is of the form

$$f + \mu h = 0,$$

the line is equally well determined, and uniquely, by any one of its points. We shall call such a line in F_4 a *syzygetic line*, since its points give on R a syzygetic pencil of quartics. The totality of syzygetic lines we shall call *the system s*, and individual lines of the system will frequently be called *s-lines*. Polar planes of syzygetic lines as to R will be called *syzygetic planes*; we denote the system of these planes by the Greek letter σ. σ-planes may be defined independently as planes which are axes of pencils of spaces which cut out of R syzygetic pencils of quartics. There are ∞^4 points in F_4; each point, in general, determines uniquely an *s*-line, but since the line may equally well be determined by any of its points, we have:

The totality of s-lines in F_4 *are a system of* ∞^3 *lines.*

Also:

There are ∞^3 σ*-planes in* F_4.

There is a unique s-line on any point.

There is a unique σ*-plane on any space.*

In order to discuss *s*-lines and the associated theory it will be necessary to recall a few known facts from the invariant theory of the binary quartic.

Let a quartic f be written:

$$f \equiv a_0 t^4 + 4 a_1 t^3 + 6 a_2 t^2 + 4 a_3 t + a_4 .$$

Its Hessian is:

$$\begin{aligned} h \equiv (a_0 a_4 - a_1^2) t^4 + 2 (a_0 a_3 - a_1 a_2) t^3 + (a_0 a_4 + 2 a_1 a_3 - 3 a_2^2) t^2 \\ + 2 (a_1 a_4 - a_2 a_3) t + a_2 a_4 - a_3^2 . \end{aligned}$$

Its cubicovariant is:

$$\begin{aligned} T \equiv\ & (a_0^2 a_3 - 3 a_0 a_1 a_2 + 2 a_1^3) t^6 + (a_0^2 a_4 + 2 a_0 a_1 a_3 - 9 a_0 a_2^2 + 6 a_1^2 a_2) t^5 \\ & + (5 a_0 a_1 a_4 - 15 a_0 a_2 a_3 + 10 a_1^2 a_3) t^4 + (- 10 a_0 a_3^2 + 10 a_1^2 a_4) t^3 \\ & + (- 5 a_0 a_3 a_4 + 15 a_1 a_2 a_4 - 10 a_1 a_3^2) t^2 \\ & + (- a_0 a_4^2 - 2 a_1 a_3 a_4 + 9 a_2^2 a_4 - 6 a_2 a_3^2) t + (- a_1 a_4 - 2 a_3^3 + 3 a_2 a_3 a_4). \end{aligned}$$

Let f have a double root. This is equivalent to saying that the point defining f is on Σ. Suppose the double root to be $t = 0$. Then

$$\begin{aligned} f &\equiv a_0 t^4 + 4 a_1 t^3 + 6 a_2 t^2, \\ h &\equiv (a_0 a_2 - a_1^2) t^4 - a_2 (2 a_1 t^3 + 3 a_2 t^2). \end{aligned}$$

In this case we have:

$$a_2 f + 2 h = (3 a_0 a_2 - 2 a_1^2) t^4. \qquad (1)$$

The vanishing of the factor $3 a_0 a_2 - 2 a_1^2$ is the condition that the other two roots of f are equal; this is significant as we shall see shortly.

It follows from (1) that if f represents a point on an osculating plane of R, the *s*-line determined by f passes through the point of osculation of this plane with R. Hence:

Every line in an osculating plane of R and through the point of osculation is an s-line.

Again, let us suppose that f is given in the canonical form

$$f \equiv a_0 t^4 + 6 a_2 t^2 + a_4 .$$

Then $a_1 = a_3 = 0$ and

$$h = a_0 a_2 t^4 + (a_0 a_4 - 3 a_2^2) t^2 + a_2 a_4 ,$$

$$T = (a_0 a_4 - 9 a_2^2)(a_0 t^5 - a_4 t).$$

Here

$$a_2 f - h = (9 a_2^2 - a_0 a_4) t^2. \tag{2}$$

The factor $9 a_2^2 - a_0 a_4$ on the right again represents by its vanishing the condition that f be the square of a quadratic.

T is the product of three quadratics, the Jacobians of the three pairs of quadratic factors of f. In our canonical form the roots of one of these quadratics are 0 and ∞. Equation (2) states the well-known fact that the square of this quadratic is in the pencil $f + \lambda h$; if $9 a_2^2 - a_0 a_4 = 0$, f and h are the same quartics and $T \equiv 0$. Now in the pencil $f + \lambda h$ occur the squares of the quadratic factors of T; any quartic which is a perfect square is represented in F_4 by a point on two osculating planes of R; in other words, a point on the double spread of Σ, γ_3. Hence:

Syzygetic lines in F_4 *are lines trisecant to* γ_3.

This may be used as a geometrical definition of *s*-lines, and is very important for our purpose.

If f is a perfect square, f and h are the same, and we have:

The points of γ_3 *fail to determine s-lines uniquely.*

This is the meaning, from our point of view, of the factors on the right in (1) and (2).

The three quadratic factors of the sextic T are apolar two and two. The theory of quadratics on R is closely associated with the spreads γ_3 and its dual g_3. A pair of points on R defines uniquely a point of γ_3, the point of intersection of the planes of R at the point. We have:

The necessary and sufficient condition that two quadratics on R *be apolar is that the line joining the points of* γ_3 *representing them be an s-line; that is, meet* γ_3 *again.*

This is an interpretation in four dimensions of the theorem that if two quadratics are apolar there is an identical relation among the squares of the quadratics and the square of their Jacobian.

We have pointed out that every space section of γ_3 is a rational quartic.

The s-lines in a space α, that is, lines trisecant to the rational quartic $\gamma_3\alpha$, are therefore lines of a quadric $s.\alpha$.

Since all lines in a plane of R and through the point of osculation are s-lines, we have:

The quadric $s.\alpha$ contains the cuspidal lines of $S\alpha$.

Consider a point x of γ_3 and a space α on it. There is a unique line through x trisecant to $\gamma_3\alpha$. Hence:

s-lines on a point x of γ_3 are on a plane π_x.

The locus of pairs of points of γ_3 on an s-line through x is a conic c_x in the plane π_x.

For each s-line through x meets this locus in two and only two points.

The following properties of the conic c_x determined on γ_3 by a point x of γ_3 are evident from the theory of binary quadratics:

c_x meets R in two points, a_{t_1} and a_{t_2}.

The planes of R at t_1 and t_2 meet at x.

x is the polar point, on π_x, of the line $a_{t_1}a_{t_2}$ as to c_x.

If x' is a point of c_x, the conic $c_{x'}$ passes through x. There are thus ∞^1 conics $c_{x'}$ through x.

The correspondence between conics c_x on γ_3 and binary quadratics on R is one-to-one.

We are not concerned here primarily with the theory of the rational quartic curve in space. Many of its properties follow from the theory which we have developed. In this connection we refer the reader to the papers by Berzolari and Marletta previously mentioned.

We point out in passing the geometrical representation of the theory of the binary quartic which we have from our point of view. Any point x of F_4 determines a binary quartic on R,

$$f = (x\,t)^4 = 0.$$

The invariants of this quartic are represented by the spreads g_2 and g_3. Σ is the discriminant. The unique s-line, p_x, through x meets γ_3 in three points x_1, x_2, x_3, defining thus the three quadratics whose product is the sextic covariant of $(x\,t)^4 = 0$. Points of p_x represent quartics of the syzygetic pencil $f + \lambda h = 0$. The polar point (along p_x) of x as to x_1, x_2, x_3 gives the Hessian of f.

There are ∞^3 s-lines in F_4, and ∞^2 planes on each; we have thus ∞^5 planes. Hence:

Planes on s-lines in F_4 are in a hypercomplex of planes, Γ.

The planes $\Gamma . \alpha$ touch the quadric $s . \alpha$; hence:

Γ *is a quadratic hypercomplex.*

Dually we have:

Lines on σ-planes in F_4 are in a quadratic hypercomplex of lines, G.

The lines $\Gamma . a\alpha$ are a quadratic complex of lines on α. On α we have also the Steiner quartic surface $\gamma_3 a \alpha$. $R a \alpha$ is an asymptotic curve on $\gamma_3 a \alpha$. The lines of $\Gamma . a \alpha$ may be regarded as the projections of s-lines from a on α; that is,

$$\Gamma . a\alpha \equiv s\,a\alpha.$$

Consider a point x of γ_3; we have a plane pencil of s-lines on x. $x a \alpha$ is a point of $\gamma_3 a \alpha$. The plane pencil of s-lines on x projects into a plane pencil of lines of $\Gamma . a \alpha$ on $x a \alpha$. Hence:

$\gamma_3 a \alpha$ is the singular surface of the complex $\Gamma . a\alpha$.

Stahl proved the existence of the complex $\Gamma . a \alpha$ from an entirely different point of view.* We omit the consideration of its properties for the sake of brevity.

Consider a line p in F_4; there is a unique s-line on every point of p. Planes on these s-lines and the line p generate the cone-spread $\Gamma . p$ determined by p, the locus of planes of Γ which contain p. p meets Σ in six points, determining six planes π_{t_i}, and hence six points a_{t_i}, of R. These six points project into the six points of inflection of $R p \pi$. Let p meet π_{t_i} in the point b_i. Then, from what we have said, $b_i a_{t_i}$ is an s-line. Hence:

The conic $\Gamma . p\pi$ is the conic on the points of inflection of $R p \pi$.

This conic may obviously be regarded as the projection from p of the points representing the Hessians of the various sets of the fundamental involution of $R p \pi$ determined by the points of p. Algebraically, if we have a quartic given by

$$\rho X_i = (\alpha_i t)^4, \tag{3}$$

and its fundamental involution is

$$(a t)^4 + \lambda (b t)^4 = 0, \tag{4}$$

and we represent by h the Hessian of the quartic (4) involving λ to the second degree, the conic on the six points of inflection of (3) is, λ being the variable parameter,

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* Stahl, *Journal für Math.*, 101, p. 307.

$$\rho X_i = |\alpha_i h|^4. \tag{5}$$

The points of the conic on the points of inflection of $Rp\pi$, which we shall call w, are thus put into one-one correspondence with the sets of the fundamental involution.

Let f be a point of p. Let the s-line, p_f, on f meet γ_3 in x_1, x_2, x_3. There are three planes on p_f which cut out conics from γ_3, the planes π_{x_1}, π_{x_2}, π_{x_3}, say. These three planes meet R in the quadratics defined by x_1, x_2, x_3,—three mutually apolar quadratics. Now any space on π_{x_1}, and in particular the unique space on π_{x_1} and p, touches γ_3 and cuts out harmonic pairs from R. We have then in π:

The three lines on a point of w which cut out from $Rp\pi$ harmonic pairs, meet $Rp\pi$ in three mutually apolar quadratics.

This is a defining characteristic of w.

If a plane π is on an s-line, $\Sigma\pi$ has three double points on a line, since s-lines are trisecant to γ_3. Hence our statement in IX, § 8.

s-lines meeting a line p are obviously on a two-way spread containing p. Any space α on p meets this two-way in p and in the two s-lines on the points in which p meets the quadric $s.\alpha$. Hence:

s-lines meeting a line p are on a cubic two-way, T_3, with p for a directrix. This T_3 meets R in the six points determining the points of inflection of $Rp\pi$.*

This T_3 meets γ_3 in a curve. Any space α on p meets this curve in six points, three on each s-line on α and meeting p. Hence:

s-lines which meet a line p meet γ_3 in a curve of order 6.

It will be obvious from the next section that the genus of a curve of order 6 on γ_3 can not be greater than one. The above curve projects from a point of F_4 into an elliptic sextic whose trisecant lines are lines of a ruled cubic surface.

The quadric spread $\Gamma.p$ meets γ_3 in a curve of order 8, out of which this sextic factors. $\Gamma.p$ meets γ_3 in an additional conic, C. Now C meets R in a pair of points defined by a point x common to γ_3 and the plane of C. C is the locus of points on γ_3 defined by pairs of points apolar to the points in which C meets R. $Cp\pi$ is the conic w of $Rp\pi$. w meets $Rp\pi$ in the six points of inflection and a further pair of points, which we shall call the points q. The points q are the projections of the points in which C meets R. We may sum this up in the theorem:

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* T_3 will be used as a generic symbol for the so-called normal cubic two-way in F_4.

Line sections of $Rp\pi$ on any point of w are apolar to the square of a quadratic which is itself apolar to q. Lines on the polar point of the line joining the points q as to w cut out of $Rp\pi$ quartics apolar to the square of q.

If the roots of q are $(mt)=0$ and $(nt)=0$, and we write

$$(pt)^2 \equiv (mt)^2 + \lambda (nt)^2,$$

then the conic w may be written parametrically

$$\rho x_i = |\alpha_i p|^2 \, |\alpha_i p'|^2,$$

where the λ in $(pt)^2$ is regarded as the varying parameter.

§ 10. *The Mapping of γ_3 on a Cutting Plane π: the Stahl Conic N of $\Sigma\pi$.*

The quartic determined on R by a point x of F_4 is:

$$x_0 t^4 + 4 x_1 t^3 + 6 x_2 t^2 + 4 x_3 t + x_4 = 0. \tag{1}$$

If this is the square of a quadratic, say

$$(a_0 t^2 + 2 a_1 t + a_2)^2 = 0, \tag{2}$$

then x is a point of γ_3. Comparing (1) and (2), we have:

$$\left.\begin{aligned} \rho x_0 &= a_0^2, \\ \rho x_1 &= a_0 a_1, \\ \rho x_2 &= \tfrac{1}{3}(a_0 a_2 + 2 a_1^2), \\ \rho x_3 &= a_1 a_2, \\ \rho x_4 &= a_2^2. \end{aligned}\right\} \tag{3}$$

If a_0, a_1 and a_2 are regarded as the homogeneous coordinates of a point in a plane π, the equations (3) define a mapping of π on γ_3 by means of a fourfold system of conics. If α_0, α_1, α_2 are line coordinates in π, we may define the system of conics

$$(\xi x) = 0$$

as all conics in π apolar to the conic

$$4\,\alpha_0 \alpha_2 - \alpha_1^2 = 0. \tag{4}$$

We shall call (4) the conic N in π. In a more geometrical mapping of π

on γ_3 to be mentioned shortly, this conic will be shown to be identical with the Stahl conic N of $\Sigma \pi$.*

N may be given parametrically:

$$a_0 = 1, \quad a_1 = -t, \quad a_2 = t^2. \tag{5}$$

The mapping defined by (3) *has no critical points; it follows that there exist no curves of odd order on γ_3; any curve of even order $2p$ on γ_3 is the map of a curve of order p in π.*

From (3) and (5) we have:

N maps into the quartic R.

Lines of π pass into conics on γ_3. The plane on this conic meets γ_3 again in the map of the polar point of the given line as to N. We have thus the ∞^2 conics c_x determined by points x of γ_3 referred to in the previous section. Spaces on the plane of a conic c_x cut out of γ_3 a varying conic $c_{x'}$; x' is on c_x. $c_{x'}$ is the map of a line on π on the polar point as to N of the line representing c_x.

The tangents to N map into conics on osculating planes of R; these conics obviously touch R at the point of osculation. *They are in fact the pure*† *second osculants of R.* For the second osculant at a point t of R is cut out of the developable of the first osculant by the plane of the osculant at t. This plane is also the plane of R at t. It follows at once that

γ_3 is the locus of pure second osculants of R.

The polar point of a line of N as to N is its point of contact. It follows that any plane of R touches γ_3 at its point of osculation, and we see again that any line in an osculating plane of R and through the point of osculation is to be regarded as an s-line. Any line in a plane of R is a bisecant line to γ_3.

The quadratic defined by a point x of γ_3 is given on N by tangents to N from the map of x on π.

Space sections of γ_3 are represented on π by conics apolar to N. Points of intersection with γ_3 of s-lines are sets of three points on π which carry a *double*

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* N is defined as the locus of the line through the three points of inflection of a cubic osculant of a quartic $Rp\pi$. Stahl, *loc. cit.*, p. 314.

† If a rational curve is given parametrically,

$$\rho x_i = (a_i t)^n,$$

we call the curves

$$\rho x = (a_i t_1)^2 (a_i t)^{n-2}$$

pure second osculants;

$$\rho x_i = (a_i t_1)(a_i t_2)(a_i t)^{n-2}$$

are *mixed* second osculants.

infinity of conics apolar to N; that is, by three-points apolar to N. The ∞^1 s-lines on a space section of γ_3 are represented by the ∞^1 triangles, inscribed in a conic apolar to N, and apolar to N.

The points in which any plane π' meets γ_3 are the maps of the common points on π of a pencil of conics all of which are apolar to N. Such a set of four points will be said to be *orthic* to N. This is the idea of the "Polviereck" of Reye. It follows that

The parameters of the nodes of $\Sigma\pi'$ *are represented on* N *by the tangents from points of a four-point orthic to* N.*

Such a set of quadratics will be called an *orthic* set.

If three quadratics are the squares of the roots of a cubic equation, the unique quadratic making up the orthic set is known to be the Hessian of the cubic. A plane trisecant to R may be considered as representing a cubic on R. Its Hessian is given by the fourth point x in which this plane meets γ_3. Its system of polar quadratics are defined by points of the unique conic on γ_3 whose plane passes through x.

Consider in F_4 a space α and a plane π on it. Let α cut out of R the points a_{t_i} with parameters t_i. We call, as before, the lines and planes of R at these points respectively p_{t_i} and π_{t_i}.

$\gamma_3\alpha$ is a rational quartic in α. $\Sigma\alpha$ is the developable of $S\alpha$ and $\gamma_3\alpha$ is its double curve. $s\,.\,\alpha$ is the unique quadric on $\gamma_3\alpha$. $s\,.\,\alpha$ contains the cusp tangents of $S\alpha$; that is, the lines $\pi_{t_i}\alpha$. Now, as we have said, the parameters t_i are a set of the fundamental involution on $\Sigma\pi$. $s\,.\,\alpha\pi$ is a conic on π, on the nodes of $\Sigma\pi$ and cutting out from $\Sigma\pi$ an additional set of four points with the parameters t_i. Hence:

Conics on the nodes of $\Sigma\pi$ *cut out of* $\Sigma\pi$ *sets of the fundamental involution on* $\Sigma\pi$.†

The quadric $s\,.\,\alpha$ contains two systems of generators. Let us call s-lines generators of the *first* system of $s\,.\,\alpha$, and lines on this quadric meeting s-lines generators of the *second* system.

Projecting $S\alpha$ from a cusp a_{t_i}, we obtain, in accordance with a former section, a cubic osculant of $\Sigma\pi$, the osculant at the point t_i. Let us call this osculant Δ_i. The three cusp tangents of Δ_i are the projections from a_{t_i} of the lines $\pi_{t_k}\alpha$, $\pi_{t_l}\alpha$, $\pi_{t_m}\alpha$. The point in which these cusp tangents meet is the intersection with π of the generator of the second system of $s.\alpha$ through a_{t_i}. The locus of such points for all spaces α on π is the Stahl conic N of $\Sigma\pi$. We have at once:

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* Meyer: "Apolarität," pp. 241, ff.

† This is essentially Stahl's proof of this fact; *Journal für Math.*, 101, p. 302.

A conic on the nodes of $\Sigma\pi$ cuts out of the Stahl conic N of $\Sigma\pi$ the same set of the fundamental involution of $\Sigma\pi$ which it cuts out of $\Sigma\pi$.

The section, $\gamma_3\alpha$, of γ_3 by any space α on π is uniformly represented on the conic $s.\alpha\pi$, the conic in which the unique quadric on $\gamma_3\alpha$ meets π, by means of the generators of the *second* system on $s.\alpha$, these generators meeting $\gamma_3\alpha$ once. We shall see that in this way γ_3 is mapped in a one-to-one way on the plane π.

Any point x of γ_3 determines a space α on x and π, and thereby a quadric $s.\alpha$ on α, and a unique point on $s.\alpha\pi$, the generator of the second kind on $s.\alpha$ on x meeting π in this point. The conics $s.\alpha\pi$ are in a pencil on π, their common points of intersection being the four nodes of $\Sigma\pi$. Now there can not be two quadrics $s.\alpha$ and $s.\alpha'$ on one conic of this pencil in π since this would require two s-lines through every point of this conic, and we know that there is a unique s-line on every point of F_4 not on γ_3. We have then:

The correspondence between γ_3 and π determined by the quadrics $s.\alpha$ on spaces α on π is one-to-one. Conics of the pencil on the four points $\gamma_3\pi$ correspond to the space sections $\gamma_3\alpha$ on π.

Further, there are no singular points of this correspondence in π; that is, there are no points of π which have no definite correspondents on γ_3. Now the space sections of γ_3 on π are represented by conics; it follows that *all* space sections of γ_3 are maps of conics on π. Hence we have:

γ_3 is mapped by our scheme from π by conics of a fourfold system; that is, by conics apolar to a definite conic, C, on π.

Let us call the pencil of conics on the nodes of $\Sigma\pi$ in π the pencil Ψ. On any conic of Ψ there is a natural three-to-one correspondence. For on any point of such a conic there is a unique s-line, p; on this s-line and π there is a space α. p meets $\gamma_3\alpha$ in three points; these three points give in turn three points on the conic of Ψ, $s.\alpha\pi$. These three points are, from what we have said in the first part of this section, apolar to the conic C of the above theorem. C is the locus of points in which two of such sets of three points on conics of the pencil Ψ coincide. Any space α on π meets R in four points a_{t_i}; the lines $\pi_{t_i}\alpha$ are s-lines on $s.\alpha$. *But these lines touch γ_3 and hence $\gamma_3\alpha$.* It follows that the generator of the second system of $s.\alpha$ on a_{t_i} meets π in a point of the conic C. Hence:

The conic C on π is the Stahl conic N of $\Sigma\pi$.

Four points on N determine four points on R. These in turn determine a space α in F_4. The quartic $\gamma_3\alpha$ is the map of the unique conic on the four points of N and apolar to N. The tangents at the four points of N determine a range of conics. Conics apolar to all conics of this range are the maps of space sections of γ_3 on the polar point a of α as to g_2. We have thus the mapping from π of the Steiner surface $\gamma_3 a\alpha'$. We point out in passing that it follows easily from this that *the involution $I_{2,2}$ determined on $\Sigma\pi$ by tangents from a point of π is defined on N by points x, y, conjugate as to the pencil Ψ.*

Let us call the generators of the second system passing through a_{t_i} of the quadric $s.\alpha$ on a space α on π, the lines l_1, l_2, l_3, l_4. l_1 is on a_{t_1} and meets $\pi_{t_2}\alpha$, $\pi_{t_3}\alpha$, $\pi_{t_4}\alpha$. π_{t_1}, π_{t_2}, π_{t_3}, π_{t_4} meet π in four points, say b_1, b_2, b_3, b_4 of the fundamental involution of $\Sigma\pi$ on $\Sigma\pi$. l_1, l_2, l_3, l_4 meet π in four points of the same set of the fundamental involution of $\Sigma\pi$ on N. Call these the points a_1, a_2, a_3, a_4. The plane $\pi_{t_1}\alpha l_2$ meets π in the line b_1a_2.

The projection of $S\alpha$ on π from a_{t_1} is the first osculant Δ_1 of $\Sigma\pi$ at the point b_1. The cusps of Δ_1 are the projections of a_{t_2}, a_{t_3}, a_{t_4}, and its cusp lines are the projections of $\pi_{t_2}\alpha$, $\pi_{t_3}\alpha$, $\pi_{t_4}\alpha$.

The line b_1a_2 in π is the cusp tangent 1 to the osculant Δ_2. The lines b_1a_2 and b_2a_1 meet in the point 12 of $g_3\alpha$, the trace of $a_{t_1}a_{t_2}$ on α. The two sets of four points a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 are on a conic Ψ on the nodes of $\Sigma\pi$. The three points b_1a_2/b_2a_1, b_2a_3/b_3a_2, b_3a_1/b_1a_3 are on a line by the Pascal theorem. This is the line 4 of the Stahl conic K of $\Sigma\pi$; it is the trace on π of the plane $a_{t_1}a_{t_2}a_{t_3}$. We may summarize what we have said in the following theorem:

A conic Ψ on the nodes of $\Sigma\pi$ cuts out of $\Sigma\pi$, and out of its Stahl conic N, a set of the fundamental $I_{1,3}$ of $\Sigma\pi$; we have thus on $\Sigma\pi$ four points b_i and on N four points a_i, these two sets of four points being ordered with respect to each other. Points of intersection of pairs of lines like b_1a_2/a_1b_2 — six points in all — are six vertices of a four-line inscribed in the cubic $g_3\pi$. The lines of this four-line touch the Stahl conic K of $\Sigma\pi$ and define on K the set of the fundamental $I_{1,3}$ of $\Sigma\pi$ which Ψ cuts out of $\Sigma\pi$ and N.

If p is a line of F_4 meeting g_3 in a point c, and the bisecant line to R through c meets R in d_{t_1} and d_{t_2}, and the plane of R at $d_{t'_1}$, say, meets p, then $Rp\pi'$ is a rational quartic with a flecnode.

The line b_1a_2 on the plane π above is such a line. But b_1a_2 is a cusp-line of a cubic osculant of $\Sigma\pi$. Hence the statement in IX, § 8.

§ 11. *Three-to-one Correspondence Determined on π by $\Sigma\pi$ and its Stahl conic N.*

It is not possible to establish a one-one correspondence between a conic in a plane and a sextic with six cusps and four nodes by means of a Cremona transformation of the plane. We have suggested by the preceding section a three-to-one correspondence in the plane π which relates N with $\Sigma\pi$ in a one-to-one way.

Spaces α on π, as we have seen, meet γ_3 in rational quartics $\gamma_3\alpha$; the quadrics $s.\alpha$ containing the quartics $\gamma_3\alpha$ meet π in conics of a pencil Ψ on the nodes a_1, a_2, a_3, a_4, say, of $\Sigma\pi$. The four points a_i are an orthic set of four points as to the conic N of $\Sigma\pi$.

Through any point x of π there is a unique s-line; there is a unique space α on this s-line and the plane π. α contains a quadric $s.\alpha$ which meets π in a conic Ψ on a_1, a_2, a_3, a_4, x. The s-line on x lies on this quadric. This s-line meets the curve $\gamma_3\alpha$ in three points. We call, as before, s-lines generators of the first system on $\gamma_3\alpha$, and lines on $\gamma_3\alpha$ which are not s-lines, generators of the second system. The three generators of the second system of $s.\alpha$ on the three points in which the s-line on x meets $\gamma_3\alpha$, meet π in three points on the conic Ψ and apolar to N. For any point x of π we determine in this way three points, y_1, y_2, y_3, on the same conic Ψ with x. There is thus determined a three-to-one correspondence of points y and x of the plane π; given a point y we have one x, for y determines a conic Ψ which in turn determines a unique space α on π; the generator of the second system on $s.\alpha$ meets $\gamma_3\alpha$ once; the s-line on $\gamma_3\alpha$ on this point meets π in the point x. Conversely, given a point x, we have three points y. This may be regarded as a three-to-one correspondence in the whole plane π or as a three-to-one correspondence on a conic Ψ. Let us examine the correspondence first from the latter point of view.

Looked upon as determined by the values of a binary parameter on Ψ the sets of points y — three-points inscribed in the conic Ψ and apolar to N — are represented by a pencil of binary cubics

$$(\alpha t)^3 + \lambda(\beta t)^3 = 0. \tag{1}$$

Such a pencil of cubics is known to be the polar cubics of a unique quartic f, and the apolar cubics of a second quartic $\bar{f}$. f and $\bar{f}$ have the same Hessian, this Hessian being the four points in which two points of a cubic of the set (1) coincide. The common Hessian of f and $\bar{f}$ is therefore given by the points of intersection of Ψ and N. We write the system of points y_i in the form

$$(f t)^3 (f t') = 0, \tag{2}$$

(2) being the polarized form of the quartic

$$(f t)^4 = 0.$$

The most general correspondence between the groups of three points (2) and a point τ of Ψ may be obtained by putting in (2)

$$t' = \frac{a\tau + b}{c\tau + d},$$

where a, b, c, d are arbitrary constants. If we take f in the canonical form

$$(f\,t)^4 \equiv f_0 t^4 + 6 f_2 t + f_4 = 0,$$

we have

$$(f\,t)^3 (f\,t') \equiv t' (f_0 t^3 + 3 f_2 t) + 3 f_2 t^2 + f_4 = 0, \qquad (3)$$

and putting

$$t' = \frac{a\tau + b}{c\tau + d},$$

(4) becomes, after putting $\tau = t$,

$$a f_0 t^4 + (b f_0 + 3 c f_2) t^3 + (3 a f_2 + 3 d f_2) t^2 + 3 (b f_2 + c f_4) t + d f_4 = 0.$$

(4) gives the points of coincidence ($\tau = t$) of our new correspondence. These four points, as is obvious, can not be assigned arbitrarily; we show that they must be apolar to $\bar{f}$, and hence must be orthic to N.* If t_1, t_2, t_3 and t_4 are the four coincidence-points of our correspondence on Ψ, and s_1, s_2, s_3 and s_4 the elementary symmetric functions of the four t's, we have, comparing the equation giving the four t's with (4), and eliminating a, b, c and d,

$$\begin{vmatrix} f_0, & 0, & 0, & 0, & 1 \\ 0, & f_0, & 3f_2, & 0, & -s_1 \\ 3f_2, & 0, & 0, & 3f_2, & s_2 \\ 0, & 3f_2, & f_4, & 0, & -s_3 \\ 0, & 0, & 0, & f_4, & s_4 \end{vmatrix} = 0,$$

or

$$(f_0 f_4 - 9 f_2^2)\,[3 f_0 f_2 s_4 - f_0 f_4 s_2 + 3 f_2 f_4] = 0. \qquad (5)$$

It is easily verified that (5) is the completely polarized form of the quartic $\bar{f}$. Hence:

Given any three-to-one binary correspondence of which the sets of three are polar cubics of a binary quartic f, the coincidence-points of the correspondence are apolar to the quartic $\bar{f}$ having the same Hessian as f.

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* Grace and Young: *Algebra of Invariants*, p. 310.

It is obvious, conversely, *that, given a quartic f on Ψ, a one-to-one correspondence between the points of Ψ and the polar cubics of f is uniquely determined by requiring that the coincidence-points of this correspondence be a given quartic apolar to $\bar{f}$.*

The quartic $\bar{f}$ is determined, in our case, on Ψ by the common tangents of N and Ψ; sets of four points whose parameters are apolar to $\bar{f}$ are four-points on Ψ and orthic to N. It is obvious that the four points a_1, a_2, a_3, a_4 orthic to N are sufficient to determine a three-to-one correspondence on any conic Ψ on a_1, a_2, a_3 and a_4, with coincidence-points at the points a_i.

We now consider this three-to-one correspondence as a correspondence in the whole plane π. Given any point x of π, we have a conic Ψ_x on x and a_1, a_2, a_3 and a_4, and determine thereby three points y_1, y_2, y_3 on the conic Ψ_x, which we may regard as correspondents of x. Again, given a y, there is a conic Ψ_y on y and a_i, and a unique point x on Ψ_y is thereby determined.

We indicate this three-to-one correspondence in π by the symbol T. We may consider T as giving two transformations of the plane π; one an x-to-y transformation, the other a y-to-x transformation. We indicate these transformations by $T\begin{pmatrix}1 & 3\\ x, & y\end{pmatrix}$ and $T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ respectively. $T\begin{pmatrix}1 & 3\\ x, & y\end{pmatrix}$ and $T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ are inverse transformations. The genus of a curve is in general invariant under $T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$, but is in general altered by $T\begin{pmatrix}1 & 3\\ x, & y\end{pmatrix}$, the correspondence between the curve and its transform being in the latter case one-to-three in general.

We represent a line, regarded as a locus of points x or y, by the symbols l_x and l_y respectively. The curve which is the locus of points corresponding to the points of l_x or l_y we shall indicate by C_y and C_x respectively.

The *order* of the correspondence T is the number of times corresponding points fall respectively on two lines l_x and l'_y of π. If this is n, it is obvious that the curves C_x and C_y are of order n, in general.

Let us call the polar lines of a_1, a_2, a_3, a_4 as to N the lines p_1, p_2, p_3, p_4. The points of intersection of p_1, p_2, p_3, p_4 we shall indicate by two-letter symbols p_1p_2, etc., and the lines joining two of the points a_1, a_2, a_3, a_4 by two-letter symbols a_1a_2, etc. If a point x is at a_1, one y is at a_1 and the other two y's are any pair of points on p_1 and apolar to N. a_1, a_2, a_3, a_4 are thus singular points of $T\begin{pmatrix}1 & 3\\ x, & y\end{pmatrix}$, and to these singular points correspond their polar lines as to N,

p_1, p_2, p_3, p_4. It follows that the curves corresponding to lines l_y in the transformation $T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ must pass once through a_1, a_2, a_3, a_4. Any conic on a_1, a_2, a_3, a_4, that is, any conic of the pencil Ψ, meets a curve C_x at the four points a_i and in the transforms by $T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ of the two points in which the line l_y meets this conic. C_x therefore meets any conic Ψ in six points and is therefore of order 3. Hence:

The three-to-one correspondence T is of order 3.

The transformations $T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ and $T\begin{pmatrix}1 & 3\\ x, & y\end{pmatrix}$ send lines into cubic curves.

We recall a few facts from a former paper* in order to discuss more fully the groups of three points y_i. The conic N and the orthic set of four points a_i determine a Desargues configuration, B, of ten lines and ten points symmetrically related. The points of this configuration are the points a_i and $p_i p_j$, and the lines are p_i and $a_i a_j$, the whole making up a symmetrical configuration of ten lines and ten points, three lines on a point and three points on a line. a_i and p_i are pole and polar as to N; similarly $a_i a_j$ and $p_i p_j$ are polar and pole as to N. If the configuration B is regarded as the section by π of the ten lines and ten planes of a complete five-point in space, the groups of points y are sets of three points, $\iota_3^{(3)}$, in which norm-curves on the five points in space meet π. The sets y or $\iota_3^{(3)}$ are groups of three points apolar to N and inscribed in conics on any four points of the configuration orthic to N. We proved, in the paper referred to, the following theorems:

a) *Two points of a set $\iota_3^{(3)}$ coincide at a point of N.*

b) *The locus of the third point of a set $\iota_3^{(3)}$ of which two points coincide is a rational sextic S with nodes at the ten points of the configuration B.*

c) *If one point y_1 of a set $\iota_3^{(3)}$ is on a line, the locus of the two other points y_2, y_3 is a quartic curve on the points of the configuration B; this quartic curve has a node on the pole, m, of the given line as to N; N is the conic on the points of tangency of tangents to this quartic from m; it passes further through the ten points, where the line joining m to any point, a_1 say, of B meets the polar line, p_1, of a_1 as to N.*

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* Conner: "Basic Systems of Rational Norm-Curves," AMERICAN JOURNAL OF MATHEMATICS, April, 1910, pp. 124 ff.

Let us call the quartic determined in this way by a line l_y the quartic $Q_y^{(1)}$. There is a unique curve $Q_y^{(1)}$ with a node at a given point. There are thus ∞^2 such curves, corresponding to the ∞^2 lines of π, when B is given.

d) *The direction of coincidence of two points of a set* $\iota_3^{(3)}$ *at a point of* N *defines a rational curve* S' *of class* 6. *This curve has for double tangents the lines of* B. *It is the polar reciprocal as to* N *of the rational sextic mentioned above with nodes at the points of* B.

A curve C_x must be rational since it is in one-one correspondence with a line l_y. We now locate its node. Let n_1 be the pole of l_y as to N. The conic on a_1, a_2, a_3, a_4, n_1 cuts out of l_y two points n_2 and n_3 such that n_1, n_2, n_3 are a group of y-points, a set $\iota_3^{(3)}$. To these corresponds one point, m, on this conic. m is the double point of C_x, the two points n_2 and n_3 of l_y giving the one point m.

Two curves C_x and C_x' meet in five variable points — four points of intersection of these cubics being fixed in the points a_i. One of these five points is the transform of the point in which l_y meets l_y'. The other four points of intersection of C_x and C_x' arise from pairs of points $y_1 y_2$, which fall respectively on l and l'. As a point y_1 runs along l, the locus of the points y_2, y_3 is a quartic curve with a node at the pole of l as to N. This curve, $Q_y^{(1)}$, meets l' in four points, thus determining four pairs $y_1^{(i)} y_2^{(i)}$ on l and l', and thence the four further intersections of C_x and C_x'.

To a point y corresponds in general a unique x; to the point $p_1 p_2$ correspond both a_1 and a_2; it follows that $p_1 p_2$ is a singular point of $T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$, and that the corresponding locus of x's passes through a_1 and a_2. This locus can be nothing but the line $a_1 a_2$. $T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ has therefore the six singular points $p_1 p_2$, etc., with the six corresponding lines $a_1 a_2$, etc. It follows that $T\begin{pmatrix}1 & 3\\ x, & y\end{pmatrix}$, *in general, sends lines into cubic curves on the six points* $p_i p_j$, *the six points of the four-line* p_i. *The curves* C_y *are a net of cubics on these six points.*

There is a unique pair of points y_1, y_2 belonging to the same set $\iota_3^{(3)}$ on any line l_y of the plane. We saw that this pair of points y transform into the node of C_x. If C_x is to have a cusp, the pair of points y_1, y_2 on the line l_y must coincide. Hence:

The locus of lines l_y *transforming by* $T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ *into cubic curves* C_x *with a cusp is the sextic line-curve* S' *of theorem* d).

It follows immediately:

The locus of cusps of cuspidal curves C_x *is the transform of* N *by* $T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$; *that is, the sextic* $\Sigma\,\pi$.

$T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ may easily be put into analytical form. Let us take as a_1, a_2, a_3, a_4 the four points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ respectively. Then N, in lines, must be of the form

$$k_0\xi_0^2 + k_1\xi_1^2 + k_2\xi_2^2 + 2\lambda(\xi_1\xi_2 + \xi_2\xi_0 + \xi_0\xi_1) = 0.$$

N is, therefore, in points,

$$\begin{aligned} N \equiv {} & (k_1k_2 - \lambda^2)x_0^2 + (k_2k_0 - \lambda^2)x_1^2 + (k_0k_1 - \lambda^2)x_2^2 \\ & + 2\lambda(\lambda - k_0)x_1x_2 + 2\lambda(\lambda - k_1)x_2x_0 + 2\lambda(\lambda - k_2)x_0x_1 = 0. \end{aligned}$$

Then

$$\begin{aligned} p_1 & \equiv (k_1k_2 - \lambda^2)y_0 + \lambda(\lambda - k_2)y_1 + \lambda(\lambda - k_1)y_2, \\ p_2 & \equiv \lambda(\lambda - k_2)y_0 + (k_2k_0 - \lambda^2)y_1 + \lambda(\lambda - k_0)y_2, \\ p_3 & \equiv \lambda(\lambda - k_1)y_0 + \lambda(\lambda - k_0)y_1 + (k_0k_1 - \lambda^2)y_2, \\ p_4 & \equiv p_1 + p_2 + p_3 \\ & \equiv (\lambda - k_1)(\lambda - k_2)y_0 + (\lambda - k_2)(\lambda - k_0)y_1 + (\lambda - k_0)(\lambda - k_1)y_2. \end{aligned}$$

$T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ is at once

$$\begin{aligned} x_0 &= (\lambda - k_1)(\lambda - k_2)\,y_0p_2p_3, \\ x_1 &= (\lambda - k_2)(\lambda - k_0)\,y_1p_3p_1, \\ x_2 &= (\lambda - k_0)(\lambda - k_1)\,y_2p_1p_2. \end{aligned}$$

The curves C_y are a net of cubic curves on the six points p_ip_j. These curves admit an infinity of point-sets $\iota_3^{(3)}$, and hence meet N with the direction of the tangent to the sextic S' of theorem d) determined by a point of intersection with N. It follows that N is the curve along which curves of the net C_y touch; that is,

N is the Jacobian curve of the net of curves C_y.

The Jacobian curve of a net of curves is also the locus of nodes of nodal curves of the net; hence:

There is a unique curve C_y with a node at a given point of N.

Our whole apparatus is obviously determinable in the reverse order. That is, we may start with four lines p_1, p_2, p_3, p_4, and a net of cubic curves C_y on the six points $p_i p_j$, the points of intersection of these four lines. The Jacobian curve of the net C_y is the four lines p_i and a conic N orthic to the four lines. Any three points which are the base-points of a pencil of the net C_y are a set $\iota_3^{(3)}$ as determined by the conic N and the orthic four-line p_1, p_2, p_3, p_4. This apparatus has eleven constants, as it should.

Consider a line l_x and its transform C_y. The six points in which C_y meets N are transforms of the six points in which l_x meets $\Sigma\pi$. Now all curves C_y have the same tangent at a point of N; if we ask for a curve C_y meeting N in a point counting twice, C_y must have a node at this point. The corresponding line l_x must meet $\Sigma\pi$ in a point counting twice; that is, must touch $\Sigma\pi$. Hence:

Curves C_y with a node are transforms of lines of $\Sigma\pi$.

Any locus of sets $\iota_3^{(3)}$ is sent by $T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ into some curve repeated three times. Conversely, to a general curve corresponds by $T\begin{pmatrix}1 & 3\\ x, & y\end{pmatrix}$ a locus of sets $\iota_3^{(3)}$.

$T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ sends a curve C_y into its correspondent line l_x taken three times. It sends conics on sets of four points like a_4, $p_2 p_3$, $p_3 p_1$, $p_1 p_2$,—another orthic four-point of B,— into lines on a_4.

The latter system of conics taken with p_4 are special curves C_y.

$T\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ sends cubics of a net on six points of B like a_1, a_2, a_3, $p_1 p_4$, $p_2 p_4$, $p_3 p_4$, — vertices of another orthic four-line of B, — into conics on a_1, a_2 and a_3.

This net of cubics is a net similarly associated with another correspondence determined by the orthic four-line $a_1 a_2$, $a_2 a_3$, $a_3 a_1$, p_4.

There are *five* orthic four-points among the ten points of our configuration B, each determining a three-to-one correspondence of points of the plane π. To deal with these five correspondences it is desirable to make a change in our notation, conforming it with that of the former paper already referred to. A configuration B, as we have said, is cut by a plane out of a complete five-point in space, and a symmetrical notation for its points and lines is given by naming the points of space 1, 2, 3, 4, 5, say, and using two-figure symbols for the points, and three-figure symbols for the lines of B. The following table indicates the desired change in the notation:

Points.		*Lines.*	
a_1	12	p_1	345
a_2	13	p_2	245
a_3	14	p_3	235
a_4	15	p_4	234
$p_1 p_2$	25	$a_1 a_2$	134
$p_2 p_3$	35	$a_2 a_3$	124
$p_3 p_1$	45	$a_3 a_1$	123
$p_1 p_4$	34	$a_1 a_4$	125
$p_2 p_4$	42	$a_2 a_4$	135
$p_3 p_4$	23	$a_3 a_4$	145

Any group of four points of B whose symbols contain the same figure, as 12, 13, 14, 15, is an orthic four-point; and the group of four lines from whose symbols the given figure is absent, as 345, 245, 235, 234, is the corresponding orthic four-line. We name the five orthic four-points of B the four-points 1, 2, 3, 4, 5, these giving the correspondences T_1, T_2, T_3, T_4, T_5 respectively.

The sets y are the same for all correspondences T_i. They are sets $\iota_3^{(3)}$ as determined by the configuration. We have thus five nets of cubics $C_y^{(i)}$; one net on the six points of each of the orthic four-lines of B. The characteristic feature of the cubics $C_y^{(i)}$ is that any curve of one of the nets admits an infinity of sets $\iota_3^{(3)}$. The curves $C_y^{(i)}$ correspond to lines in the transformation $T_i\begin{pmatrix}1 & 3\\ x, & y\end{pmatrix}$. N is the common Jacobian of all of these nets. Two cubics of the same net meet in three variable points — a set $\iota_3^{(3)}$; two cubics from different nets meet in *two* sets $\iota_3^{(3)}$.

To a given group, y_1, y_2, y_3, correspond by $T_i\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$ five points, x_1, x_2, x_3, x_4, x_5. The correspondence between any two of these points, say x_1 and x_5, is one-to-one. x_1 determines by $T_1\begin{pmatrix}1 & 3\\ x, & y\end{pmatrix}$ a unique set $\iota_3^{(3)}$ which in turn, by $T_5\begin{pmatrix}3 & 1\\ y, & x\end{pmatrix}$, determines a unique point x_5. Similarly, x_1 is unique when x_5 is given. Hence between the points x_1 and x_5 we have a Cremona correspondence;

there is a Cremona transformation T_{15} which sends x_1 into x_5, its inverse T_{51} sending x_5 into x_1.

T_{15} is the product of $T_1\binom{1\ \ 3}{x,\ y}$ and $T_5\binom{3\ \ 1}{y,\ x}$; thus

$$T_{15} = T_1\binom{1\ \ 3}{x,\ y}\,T_5\binom{3\ \ 1}{y,\ x}.$$

Now $T_1\binom{1\ \ 3}{x,\ y}$ sends a line l_x into a cubic curve $C_y^{(1)}$. $T_5\binom{3\ \ 1}{y,\ x}$ sends a cubic $C_y^{(1)}$ into a conic on 25, 35, 45. Hence T_{15} sends the line l_x into a conic on 25, 35, 45. Similarly T_{51} sends a line into a conic on 12, 13, 14. 15 is a fixed point for the correspondences T_1 and T_5, and hence is a fixed point for T_{15}. Hence:

T_{15} is a quadratic Cremona transformation having 12, 13, 14 *as singular points, with* 25, 35, 45 *as singular points of the inverse transformation, T_{51}. The transformation is completely defined by these facts and the additional condition that* 15 *be a fixed point.*

When the singular triangles and a pair of corresponding points are given for a quadratic Cremona transformation, a linear construction for the correspondence is known. This gives a geometrical determination of our correspondence $T_1\binom{1\ \ 3}{x,\ y}$. Choose any other correspondence, say T_5. Let x' be the transform of x by T_{15}. The conic on x', 15, 25, 35, 45, meets the conic on x, 15, 12, 13, 14, in the point 15 and in a set $\iota_3^{(3)}$ which corresponds to x and x' in $T_1\binom{1\ \ 3}{x,\ y}$ and in $T_5\binom{1\ \ 3}{x,\ y}$ respectively. The linear construction of x when a corresponding y is given seems too complicated to be of much interest.

§ 12. *Further Theory of Quadratics on R. Curves Determined by a Quadratic on R p π.*

We saw in § 9 that the theory of quadratics on R is closely associated with that of the spread γ_3; dually we should expect the spread g_3 to be closely connected with quadratics on R. We shall point out in this section just what this relation is, and use the theory thus developed to show the existence of certain curves determined by quadratics on $R p \pi$. In the following section we shall use the same facts in the discussion of the nodes of $R p \pi$.

Any quartic on R determines a point in F_4. This is projected from a line p into a point in a plane π; this point is covariantly associated with the corresponding quartic on $Rp\pi$. The determination of this covariant point obviously fails when the given quartic on R is a set of the fundamental involution of $Rp\pi$.

Conversely, given a point in π, there is a plane on p and this point, and hence there is an involution of ∞^2 quartics on R with which a point of π is associated in this manner. We have seen the way in which the coordinates of the point in π are determined when the binary quartic on R is given; if $Rp\pi$ is given by

$$\rho x_i = (\alpha_i t)^4, \tag{1}$$

and the given binary quartic is

$$(\mu t)^4 = 0,$$

then the point is

$$\rho x_i = |\alpha_i \mu|^4, \tag{2}$$

the definition of the point in π being merely that the sections of $Rp\pi$ by lines on it are apolar to the quartic μ. If

$$(f t)^4 + \lambda (f' t)^4 = 0$$

is the fundamental involution of $Rp\pi$, then (2) is equally well defined by any quartic of the system

$$(\mu t)^4 + \lambda (f t)^4 + \lambda' (f' t)^4 = 0.$$

If the given quartic $(\mu t)^4 = 0$ is the square of a quadratic, the point in F_4 which it defines is on γ_3. We have seen that the locus of points on γ_3 which define quadratics apolar to a given quadratic

$$(k t)^2 = 0 \tag{3}$$

is a conic, C_k, which meets R in the pair of points (3) and whose plane passes through the point on γ_3 defining (3), the point k say. Tangents from k to C_k touch at the points (3) of R. Let the pencil of quadratics apolar to $(kt)^2 = 0$ be

$$(p t)^2 = (a t)^2 + \lambda (b t)^2. \tag{4}$$

The coefficients of $(pt)^2$ involve the parameter λ linearly. From (4) we have

$$(k t)^2 \equiv |a b| (a t)(b t).$$

C_k projects from p into the conic $C_k p\pi$, on π. $C_k p\pi$ is, in terms of the parameter λ of (4),

$$\rho x_i = |\alpha_i p|^2 |\alpha_i p'|^2. \tag{5}$$

(5) is a conic on π covariantly associated with the quadratic $(kt)^2 = 0$ on $Rp\pi$.

It meets $Rp\pi$ in the pair of points $(kt)^2 = 0$. The pole of the line joining this pair of points as to $C_k p\pi$ is the point

$$\rho x_i = |\alpha_i k|^2 |\alpha_i k'|^2,$$

the point of π such that sections of $Rp\pi$ through it are apolar to the square of the quadratic k.

There exist on the cubic spread g_3 a system of ∞^2 lines which are not bisecant to R, and which we may call after Segre, *axes* of g_3. There are three axes of g_3 on any space α; $g_3\alpha$ is a cubic surface with the four nodes $R\alpha$; it contains the six lines joining the four nodes, and three others lying on a plane, the unique σ-plane on α. The latter lines are axes of g_3.

Bisecant lines to R meeting an axis pg of g_3 are on a two-way T_3, — of order 3, since this two-way is met by any space on the given axis pg in pg itself and in a pair of lines bisecant to R.* R meets the generators of this T_3 twice, and does not meet its directrix pg. The lines of T_3 meet R in pairs of an involution on R, or in quadratics apolar to a given quadratic on R, this quadratic being determined by a pair of tangents to R meeting pg. There is thus a unique axis of g_3, pg, meeting two tangent lines of R, — two lines of the spread of tangents S, — and the bisecant lines to R meeting pg define on R the pencil of quadratics apolar to the pair of points of tangency of the two tangent lines. A space bitangent to R meet g_3 in a ruled cubic surface with the line joining the pair of points of tangency as a double line; the axis thus determined is the directrix of this ruled surface.

An interesting configuration of lines is determined on g_3 by three mutually apolar quadratics on R. Any point, a_1, of g_3 determines, by the bisecant line to R through it, a quadratic q_1 on R. Through a_1 there pass two axes of g_3, p_2 and p_3. Bisecant lines to R meeting p_2 and p_3 define on R the involutions of quadratics apolar to two further quadratics, q_2 and q_3. In particular, q_1 is apolar to q_2 and q_3. On the plane $p_2 p_3$ there is a third axis of g_3, p_1. p_1, p_2 and p_3, three axes of g_3 on a σ-plane, meet in three points, a_1, a_2, a_3. These define by their bisecants to R three mutually apolar quadratics q_1, q_2, q_3. It is evident from what we have said that tangents to R at the two points q_1 meet a_1, etc.

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* Cf. Segre, *loc. cit.*

The spread, Σ, of osculating planes of R, is a linear combination of g_2^3 and g_3^2, and hence can meet g_3 only in the spread

$$g_2 = g_3 = 0, \tag{6}$$

taken three times. Now (6) is the spread S of tangents to R. Hence:

$g_3 \pi$ passes through the cusps of $\Sigma \pi$ with the cusp tangents of $\Sigma \pi$.

If $g_3 \pi$ is three lines, π is a σ-plane, and we have:

The necessary and sufficient condition that $\Sigma \pi$ be a quartic projectively dual to the lemniscate is: a) $g_3 \pi$ is three lines; or b) The fundamental involution of $\Sigma \pi$ is of the form $f + \lambda h = 0$, — a syzygetic pencil of quartics.

Dually:

R projects from an s-line, p, into a quartic curve $Rp\pi$ with three biflecnodes.

If π is on an axis of g_3, the curve $\Sigma \pi$ has a singularity which is the dual of a biflecnode. $g_3 \pi$ is then a line and a conic. If π contains another axis of g_3, it must contain two. Hence, dually:

If a quartic $Rp\pi$ have a biflecnode and an additional flecnode, it must have three biflecnodes; that is, it must be a projection, real or imaginary, of the lemniscate.

All planes on an s-line are planes of the hypercomplex Γ of § 9. Hence:

The conic, w, on the points of inflection of a quartic with three biflecnodes vanishes identically. This condition is necessary and sufficient for three biflecnodes.

We have seen that any axis of g_3 determines a ruled cubic two-way T_3 whose generators are lines bisecant to R and which mark out on R pairs of an involution of quadratics apolar to a quadratic k on R. The lines of any T_3 in F_4 project from a line into the lines of a rational curve of class 3. The directrix of T_3 projects into the double line of this curve. Hence we have:

Lines joining pairs of an involution, — quadratics apolar to a quadratic k, — on $Rp\pi$ touch a rational curve ρ_k of class 3.

This theorem may easily be proved analytically in the plane. Let $Rp\pi$ be

$$\rho x_i = (\alpha_i t)^4.$$

The line joining the two points t_1, t_2 is

$$\begin{aligned}\rho \xi_i = |\alpha_k \alpha_l| \{ (\alpha_k t_1)^3 (\alpha_l t_2)^3 + (\alpha_k t_1)^2 (\alpha_k t_2) (\alpha_l t_1) (\alpha_l t_2)^2 \\ + (\alpha_k t_1) (\alpha_k t_2)^2 (\alpha_l t_1)^2 (\alpha_l t_2) + (\alpha_k t_2)^3 (\alpha_l t_1)^3 \} \\ (i, k, l = 1, 2, 3;\ 2, 3, 1;\ 3, 1, 2).\end{aligned} \tag{7}$$

If t_1 and t_2 are defined by

$$(a\,t)^2 + \lambda\,(b\,t)^2 = 0, \tag{8}$$

the symmetric functions $t_1 t_2$, $t_1 + t_2$ and 1 are given as linear functions of λ by (8), and these substituted in (7) give the ξ's as cubic functions of λ.

We now find the double line of the cubic $\rho_k^{(3)}$. The directrix p_k of the T_3 of bisecant lines to R joining pairs of points apolar to k projects into this double line. Let $(m\,t)$ and $(n\,t)$, or simply m and n, be the roots of $(k\,t)^2 = 0$. Then, in accordance with the notation above,

$$m\,n \equiv (k\,t)^2 = |a\,b|\,(a\,t)\,(b\,t).$$

The tangents to R at m and n meet p_k. It is easy to verify that the points of p_k define on R quartics of the form

$$(l\,t)^4 \equiv m\,n\,(\mu_1 m^2 + \mu_2 n^2) = 0. \tag{9}$$

The double line of $\rho_k^{(3)}$ is then

$$\rho\,x_i = |\alpha_i\,l\,|^4, \tag{10}$$

where μ_1/μ_2 is the variable parameter. The quartic

$$(l\,t)^4 = m\,n\,(\mu_1 m^2 + \mu_2 n^2)$$

has the neutral pair

$$\mu_1 m^2 - \mu_2 n^2 = 0. \tag{11}$$

The form (9) may be determined directly from the pencil

$$(a\,t)^2 + \lambda\,(b\,t)^2 = 0.$$

There are two values of λ for which this is a perfect square; these are given by

$$(a_0 + \lambda\,b_0)\,(a_2 + \lambda\,b_2) - (a_1 + \lambda\,b_1)^2 = 0.$$

Let the roots of this be λ_1 and λ_2. Then

$$(a\,t)^2 + \lambda_1\,(b\,t)^2 \equiv m^2,$$
$$(a\,t)^2 + \lambda_2\,(b\,t)^2 \equiv n^2.$$

(11) must be of the form $(a\,t)^2 + \lambda\,(b\,t)^2$. We write then

$$(a\,t)^2 + \lambda\,(b\,t)^2 \equiv \mu_1 m^2 - \mu_2 n^2 = \mu_1\,[(a\,t)^2 + \lambda_1\,(b\,t)^2] - \mu_2\,[(a\,t)^2 + \lambda_2\,(b\,t)^2].$$

Hence:

$$\mu_1 - \mu_2 = 1, \qquad \mu_1 \lambda_1 - \mu_2 \lambda_2 = \lambda,$$

and we can take, to within a factor,

$$\mu_1 = \lambda - \lambda_2, \qquad \mu_2 = \lambda - \lambda_1,$$

and (9) becomes

$$\begin{aligned}(l\,t)^4 = |a\,b|\,(a\,t)\,(b\,t)\,.\,[\,|a\,b|^2.(a\,t)^2 - |a\,a'|^2.(b\,t)^2 \\ + \lambda\,\{\,|b\,b'|^2.(a\,t)^2 - |a\,b|^2.(b\,t)^2\}].\end{aligned} \tag{12}$$

The λ in (12) is the same as the λ in (7) and (8).

Through any point, a_1, of g_3 we have a unique bisecant line to R, and two axes of g_3. There are thus determined, as we have seen, three mutually apolar quadratics, q_1, q_2, q_3, on R. But we have the additional theorem:

The quartic defined on R by the point a_1 of g_3 is the product $q_2 q_3$. Its neutral pair is q_1.

§ 13. *The Nodes of $Rp\pi$.*[*]

A line p meets g_3 in three points, thus determining three lines bisecant to R and meeting p. Projecting from p, the three quadratics so determined are the nodes of $Rp\pi$. Associated with every node of $Rp\pi$ is a bisecant line to R, and a catalectic set of the fundamental involution of $Rp\pi$ determined by the point in which this bisecant to R meets π.[†]

To use the notation of the previous section, let us call the point in which a bisecant line to R meets p $a_1^{(1)}$. This bisecant meets R in two points defined by a quadratic $q_1^{(1)}$. There are two axes of g_3 through $a_1^{(1)}$; let us call these $p_2^{(1)}$ and $p_3^{(1)}$. We have then the configuration of § 12, which is determined uniquely by any point $a_1^{(1)}$ of g_3: three lines of g_3 on a syzygetic plane in F_4, the three vertices $a_1^{(1)}$, $a_2^{(1)}$, $a_3^{(1)}$ defining by their bisecants to R three mutually apolar quadratics. Let us consider for a moment the line $p_3^{(1)}$ through $a_1^{(1)}$. Bisecant lines to R which meet $p_3^{(1)}$ lie on a cubic two-way T_3 of which $p_3^{(1)}$ is the directrix. Spaces on p and on $p_3^{(1)}$ cut out of this T_3 an involution of generators;[‡] the spaces on the double lines of this involution meet R in two points, each counted twice; that is, they are spaces on p bitangent to R. Hence $p_3^{(1)}$ is a line on two bitangent spaces to R from p; similarly $p_2^{(1)}$ is a line on the other two bitangent spaces to R from p. Again, the set of the fundamental involution of $Rp\pi$ defined by the point $a_1^{(1)}$ of p is given on R by the two pairs of tangent lines meeting $p_2^{(1)}$ and $p_3^{(1)}$ (§ 12). We have, therefore, from the theory of § 12:

Associated with a node of $Rp\pi$ is a definite catalectic set of the fundamental involution. The node determines a definite pairing off of the double tangents 1, 2, 3, 4 of $Rp\pi$, say 12, 34. The catalectic set of the fundamental involution associated with the node is given by the tangents to $Rp\pi$ (not double tangents) from 12 and 34. The quadratic giving the node is the neutral pair of this set.

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[*] Cf. Marletta, *Annali di Mat.*, *loc. cit.*, p. 111.

[†] Stahl, *Math. Annalen*, 38, p. 571.

[‡] Veronese, *Math. Annalen*, 19, p. 231.

Again:

Lines on a point 12 *cut out from* $Rp\pi$ *two quadratics, each of which is apolar to the points defined by the pair of tangents from* 12. *The line* 12, 34 *cuts out of* $Rp\pi$ *a quartic of a syzygetic pencil, of which the sextic covariant is the product of the nodal parameters and the associated set of the fundamental involution. Lines joining pairs of points whose parameters are apolar to the parameters of the node touch a rational curve of class* 3 *having* 12/34 *for double line. Two points of the diagonal triangle of sets of the syzygetic pencil of quartics on* $Rp\pi$ *are at* 12 *and* 34; *that is, this syzygetic pencil is cut out of* R *by pairs of lines on either* 12 *or* 34. *The correspondence thus determined in either of these pencils of lines is* (2, 2).

§ 14. *The Fundamental Involution of* $Rp\pi$ *in Lines, or of* $\Sigma\pi$ *in Points.*

Points of π determine on $Rp\pi$, by means of tangents on them, an involution $I_{2,4}$ of binary sextics. The involution $I_{3,3}$ of all sextics apolar to sets of this $I_{2,4}$ is easily determinable algebraically when the fundamental involution of $Rp\pi$ is given.

Let us consider a rational quartic $Ra\alpha$ in space. The surface $\gamma_3 a\alpha$ is the locus of pure second osculants of $Ra\alpha$, or is the locus of planes in α cutting out harmonic pairs from $Ra\alpha$. The four stationary points of $Ra\alpha$ are the projections from a of the four points

$$(a\,t)^4 = 0$$

defined on R by a. All plane sections of $Ra\alpha$ are apolar to $(a\,t)^4$. The sextic covariant T of $(at)^4$ is the three pairs of points on $Ra\alpha$ such that the osculating plane at each point of the pair passes through the other. The chords of $Ra\alpha$ through these pairs of points meet in a point, the point $s\,.\,a\alpha$, the trace on α of the unique s-line on a. $s\,.\,a\alpha$ is the triple point of $\gamma_3 a\alpha$. The three chords are the double lines of $\gamma_3 a\alpha$.*

$Sa\alpha$ is the developable of tangents to $Ra\alpha$. It is of order 6. Any linear complex in α contains six lines of $Sa\alpha$, thus determining a binary sextic on $Ra\alpha$, the points of contact of the six lines of $Sa\alpha$. A linear complex being in general determined by five lines, it follows that the six points of contact of lines on $Sa\alpha$ that lie in a linear complex are in an involution $I_{5,1}$; that is, they are apolar to a definite sextic on $Ra\alpha$.† This sextic must be a covariant of $(at)^4$, and can be nothing but the covariant T. In particular, we have:

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* Marletta, *Annali di Mat.*, *loc. cit.*, p. 102.

† Study, *Leipz. Berichte*, 1886, p. 7.

Lines of $R\,a\,\alpha$ which meet a given line in α determine on $R\,a\,\alpha$ six parameters apolar to T.

Let us project from any point b of α on a plane π on α, thus obtaining a rational plane quartic $R\,a\,b\,\pi$. We have from the above theorem:

Lines of $R\,a\,b\,\pi$ on a point of π determine on $R\,a\,b\,\pi$ six parameters apolar to the sextic covariant T of $(a\,t)^4$.

Now $(a\,t)^4$ is *any* set of the fundamental involution of $R\,a\,b\,\pi$. It follows that:

Lines of a rational quartic $R\,p\,\pi$ on a point of π determine on $R\,p\,\pi$ a set of six parameters apolar to all sextic covariants T of sets of the fundamental involution of $R\,p\,\pi$.

The sextic covariant of a quartic

$$a + \lambda\, b = 0$$

is of the form

$$T(a) + \lambda\, M(a\,b) + \lambda^2 M(b\,a) + \lambda^3\, T(b) = 0, \tag{1}$$

where we indicate by $T(a)$, $T(b)$ the sextic covariants of a and b respectively and M is a symbol for an intermediate covariant of a and b determined from T.

The four forms which are the coefficients of (1) *generate the fundamental involution $I_{3,\,3}$ of $R\,p\,\pi$ taken in lines, since these four forms must be linearly independent.*

§ 15. *The Curves $S\,a\,p\,\pi$ and $\Sigma\,\pi'\,p\,\pi$.*

The projections from a line p of space sections of S include first osculants of $R\,p\,\pi$ as special cases, as we have seen (§ 3); similarly, the projections from a line p of plane sections of Σ include second osculants of $R\,p\,\pi$ as special cases. The space of R at any point meets S in the first osculant of that point, and the plane of R at any point meets Σ in the pure second osculant of that point.

The following theorems as to curves traced on S are sufficiently obvious:

Any curve, C, on S meeting the lines of S once is perspective to S and is rational. $C\,p\,\pi$ is perspective to $R\,p\,\pi$ (taken in lines).

Any curve, K, on S touches all the planes of R and touches each plane of R as often as it meets a line of S.

For a plane of R contains two consecutive lines of S.

Let K meet each line of S k times. Then we have:

The curve $K\,p\,\pi$ has the stationary lines of $R\,p\,\pi$ as k-fold tangents.

For instance, a general quadric of F_4 meets S in a curve K of order 12; K meets each line of S twice. The curve $K\,p\,\pi$ is a 12-ic curve on π having the stationary lines of $Rp\pi$ for double tangents and having eight cusps at the projections of the points in which the given quadric spread meets R.

Any curve drawn on S osculates the spaces of R as often as it meets the lines of S.

For a space of R contains three consecutive tangents of R.

A space α meets S in a rational curve of class 4 and order 6. $S\alpha$ meets each line of S once. Hence:

The curves $S\alpha p\pi$ are perspective to $Rp\pi$ taken in lines. They have cusps at the projections of the four points where α meets R. They are rational curves of class 6 *and order* 6 ; *they have four cusps and four stationary lines, six nodes and six double tangents. They touch the six stationary lines of R. A curve $S\alpha p\pi$ is uniquely determined on π when $Rp\pi$ and the quartic*

$$(\alpha\, t)^4 = 0$$

defining its cusps, are given.

If $Rp\pi$ is

$$\rho\, x_i = (\alpha_i\, t)^4$$

and the quartic giving the cusps of $S\alpha p\pi$ is

$$(\alpha\, t)^4 = 0,$$

then $S\alpha p\pi$ is

$$\rho\, x_i = |\alpha_i\, \alpha|\, (\alpha_i\, t)^3\, (\alpha\, t)^3.\text{*}$$

Let the space α meet p in the point a. Then $(at)^4$ is the unique set of the fundamental involution of $Rp\pi$ which is apolar to $(\alpha t)^4$. $R\alpha\beta$ is a rational quartic in the space β. Let π be chosen on β and let p meet β in b. Then $R\alpha\beta p\pi$ is $Rp\pi$. $S\alpha a\beta b\pi$ is $S\alpha p\pi$. $S\alpha a\beta$ is the section by the plane $a\beta$ of the developable, $Sa\beta$, of $Ra\beta$. The points $(at)^4 = 0$ are the stationary points on $Ra\beta$, and the tangents at these points are inflectional generators on the developable of $Ra\beta$. We have, then:

The quartic $(\alpha\, t)^4$ on $Rp\pi$ determines a unique set, $(at)^4 = 0$, of the fundamental involution of $Rp\pi$ to which it is apolar. The tangents to $Rp\pi$ at the points $(a\, t)^4 = 0$ pass through the four points of inflection of $S\alpha p\pi$.

We omit for the sake of brevity the consideration of the specializations of the curve $S\alpha p\pi$ when the quartic α is specially chosen. Reductions in the order of $S\alpha p\pi$ occur for coincidences among the roots of α; if all four roots coincide at $t=\tau$, we have the first osculant of $Rp\pi$ at the point τ. The above theorems may be stated for first osculants; they are known, but they serve to show the close analogy between first osculants and the curves with which we are dealing.

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* § 3.

We may state the further theorems, omitting the proof, which is easy:

The six nodes of $S\alpha p\pi$ *are on a conic.*

The six double tangents of $S\alpha p\pi$ *touch a conic.*

Let π', a plane in F_4, be the axis of the pencil of spaces

$$(\alpha x)+\lambda(\beta x)=0.$$

Then $\Sigma\pi' p\pi$ is a rational line-quartic on π associated with the pencil of binary quartics on $R\pi p$:

$$(\alpha t)^4+\lambda(\beta t)^4=0.$$

The curves $S[\alpha+\lambda\beta]p\pi$ pass through the six cusps of $\Sigma\pi' p\pi$. $\Sigma\pi' p\pi$ is given in points (§ 3) by

$$\rho X_i=|\alpha_i\alpha|\,|\alpha\beta|\,|\beta\alpha_i|\,(\alpha t)^2(\beta t)^2(\alpha_i t)^2.$$

JOHNS HOPKINS UNIVERSITY, *April*, 1910.